U64. Let x be a real number. Define the sequence $(x_n)_{n\geq 1}$ recursively by $x_1 = 1$ and $x_{n+1} = x^n + nx_n$, $n \geq 1$. Prove that

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right) = e^{-x}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Arkady Alt, San Jose, California, USA

From $x_{n+1} = x^n + nx_n$ we get $\frac{x_{n+1}}{n!} = \frac{x^n}{n!} + \frac{x_n}{(n-1)!}$, for $n \geq 1$. Then

$$\sum_{k=1}^{n}\frac{x^k}{k!} = \sum_{k=1}^{n}\left(\frac{x_{k+1}}{k!} - \frac{x_k}{(k-1)!}\right) = \frac{x_{n+1}}{n!} - \frac{x_1}{0!} = \frac{x_{n+1}}{n!} - 1,$$

yielding

$$\frac{x_{n+1}}{n!} = \sum_{k=0}^{n}\frac{x^k}{k!}.$$

It follows that

$$\prod_{k=1}^{n}\left(1 - \frac{x^k}{x_{k+1}}\right) = \prod_{k=1}^{n}\left(\frac{x_{k+1} - x^k}{x_{k+1}}\right) = \prod_{k=1}^{n}\frac{kx_k}{x_{k+1}} = n!\prod_{k=1}^{n}\frac{x_k}{x_{k+1}} = \frac{n!}{x_{n+1}}.$$

Thus

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right) = \lim_{n\to\infty}\frac{n!}{x_{n+1}} = \frac{1}{\lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!}} = \frac{1}{e^x} = e^{-x}.$$

Second solution by Brian Bradie, Christopher Newport University, USA Using the recurrence relation, we find

$$\begin{aligned}\prod_{n=1}^{N}\left(1 - \frac{x^n}{x_{n+1}}\right) &= \prod_{n=1}^{N}\frac{x_{n+1} - x^n}{x_{n+1}} \\ &= \prod_{n=1}^{N}\frac{nx_n}{x_{n+1}} = \frac{x_1}{x_2}\cdot\frac{2x_2}{x_3}\cdot\frac{3x_3}{x_4}\cdot\cdots\cdot\frac{Nx_N}{x_{N+1}} \\ &= \frac{N!}{x_{N+1}}. \qquad (1)\end{aligned}$$

Next, we establish that

$$x_n = (n-1)!\sum_{k=0}^{n-1}\frac{x^k}{k!}. \qquad (2)$$