Undergraduate problems

U49. Let $f : [0,1] \to [0,\infty)$ be an integrable function. Prove that

$$\int_0^1 f(x)dx \cdot \int_0^1 x^3 f(x)dx \geq \int_0^1 xf(x)dx \cdot \int_0^1 x^2 f(x)dx.$$

Proposed by Cezar Lupu, Bucharest and Mihai Piticari, Campulung, Romania

First solution by Arkady Alt, California, USA.

Let

$$R = \int_0^1 f(x)dx \cdot \int_0^1 x^3 f(x)dx = \int_0^1 \int_0^1 f(x)f(y)x^3 dxdy = \int_0^1 \int_0^1 f(x)f(y)y^3 dxdy$$

and

$$L = \int_0^1 xf(x)dx \cdot \int_0^1 x^2 f(x)dx = \int_0^1 \int_0^1 f(x)f(y)x^2 ydxdy = \int_0^1 \int_0^1 f(x)f(y)y^2 xdxdy.$$

Since $x^3 + y^3 \geq x^2 y + y^2 x$ then

$$\begin{aligned}2R = &\int_0^1 \int_0^1 f(x)f(y)x^3 dxdy + \int_0^1 \int_0^1 f(x)f(y)y^3 dxdy = \\ &\int_0^1 \int_0^1 f(x)f(y)(x^3+y^3)dxdy \geq \int_0^1 \int_0^1 f(x)f(y)(x^2y+y^2x)dxdy = \\ &\int_0^1 \int_0^1 f(x)f(y)x^2 ydxdy + \int_0^1 \int_0^1 f(x)f(y)y^2 xdxdy = 2L \Leftrightarrow R \geq L.\end{aligned}$$

Second solution by Li Zhou, Polk Community College, Winter Haven

Let

$$D = \int_0^1 f(x)dx \cdot \int_0^1 x^3 f(x)dx - \int_0^1 xf(x)dx \cdot \int_0^1 x^2 f(x)dx.$$

Then

$$\begin{aligned}D &= \int_0^1 \int_0^1 x^3 f(x)f(y)dxdy - \int_0^1 \int_0^1 x^2 yf(x)f(y)dxdy \\ &= \int_0^1 \int_0^1 (x-y)x^2 f(x)f(y)dxdy.\end{aligned}$$