U410. Let a, b, c be real numbers such that $a+b+c=5$. Prove that

$$\left(a^2+3\right)\left(b^2+3\right)\left(c^2+3\right) \geq 192.$$

Proposed by Marius Stănean, Zalău, România

Solution by Arkady Alt, San Jose, CA, USA

Due to symmetry we can assume that $c := \min\{a, b, c\}$ .Then $c \leq \dfrac{5}{3}$ and $a+b \geq \dfrac{10}{3}$.

Let $p := a+b$. Since $(a^2+3)(b^2+3) \geq 3(a+b)^2 \iff (ab-3)^2 \geq 0$ suffice to prove the inequality

$$3(a+b)^2(c^2+3) \geq 192 \iff (a+b)^2(c^2+3) \geq 64$$

or, in more compact notation, prove

$$p^2\left(\left((5-p)^2+3\right)\right) \geq 64 \iff p^2\left(p^2-10p+28\right) \geq 64.$$

Since $p \geq \dfrac{10}{3}$ we have

$$p^2\left(p^2-10p+28\right)-64 = p^4-10p^3+28p^2-64 =$$

$$\left(p^2-2p-4\right)(p-4)^2 \geq 0 \text{ (because } p^2-2p-4 = (p-1)^2-5 \geq \left(\frac{10}{3}-1\right)^2-5 = \frac{4}{9} > 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Prajnanaswaroopa S, Bangalore, Karnataka, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Alessandro Ventullo, Milan, Italy; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Albert Stadler, Herrliberg, Switzerland.