U357. Evaluate

$$\lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right]^n$$

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Let $a_n := \ln\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)...\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n = n\left(\sum_{k=1}^n \ln\left(1+\frac{k}{n^2}\right)-\frac{1}{2}\right), n\in\mathbb{N}.$

Since $\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}+o\left(x^3\right)$ then $\ln\left(1+\frac{k}{n^2}\right) = \frac{k}{n^2}-\frac{k^2}{2n^4}+\frac{k^3}{3n^6}+o\left(\frac{1}{n^3}\right)$ for $k=1,2,...,n$.

Hence,

$$a_n = n\left(\sum_{k=1}^n\left(\frac{k}{n^2}-\frac{k^2}{2n^4}+\frac{k^3}{3n^6}+o\left(\frac{1}{n^3}\right)\right)-\frac{1}{2}\right) =$$

$$\sum_{k=1}^n\left(\frac{k}{n}-\frac{k^2}{2n^3}+\frac{k^3}{3n^5}+no\left(\frac{1}{n^3}\right)\right)-\frac{n}{2} = \sum_{k=1}^n\frac{k}{n}-\frac{n}{2}-\sum_{k=1}^n\frac{k^2}{2n^3}+\sum_{k=1}^n\frac{k^3}{3n^5}+o\left(\frac{1}{n^2}\right) =$$

$$\frac{n(n+1)}{2n}-\frac{n}{2}-\frac{n(n+1)(2n+1)}{12n^3}+\frac{n^2(n+1)^2}{12n^5}+o\left(\frac{1}{n^2}\right) =$$

$$\frac{1}{2}-\frac{(n+1)(2n+1)}{12n^2}+o\left(\frac{1}{n}\right)$$

and, therefore, $\lim_{n\to\infty} a_n = \frac{1}{2}-\frac{1}{6} = \frac{1}{3}$.

Since

$$\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)...\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n = e^{a_n}$$

then

$$\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)...\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n = e^{\lim\limits_{n\to\infty} a_n} = \sqrt[3]{e}.$$

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