U284. Let $a_n = \left\{\sqrt{n^2+1}\right\}$ be the sequence of real numbers, where $\{x\}$ denotes the fractional part of x. Find $\lim_{n\to\infty} na_n$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Arkady Alt , San Jose ,California, USA

Since $n^2 < n^2+1 < (n+1)^2$, we have that $\left\lfloor \sqrt{n^2+1} \right\rfloor = n$; hence

$$a_n = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n}.$$

This gives

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}+n} = \frac{1}{2}.$$

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Pamplona, Navarra, Spain; Stanescu Florin Serban Cioculescu Gaesti school, Dambovita, Romania; Li Zhou, Polk State College, FL, USA; Nikolaos Kolliopoulos, National Technical University of Athens, Greece; Konstantinos Tsouvalas University of Athens, Greece; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Qafqaz University Problem Solving Group; Corneliu Mănescu-Avram, "Henri Mathias Berthelot" Secondary School, Ploieşti, Romania; Daniel Vacaru, Colegiul Economic Maria Teiuleanu, Pitesti; Daniele Di Tullio, Università di Roma "Tor Vergata", Roma, Italy; David Rose, Polk State College, FL, USA; Mathematical Group "Galaktika shqiptare", Albania; Michel Faleiros Martins, Instituto Tecnológico de Aeronáutica, Brazil; Michelle Andersen, College at Brockport, SUNY; Nikolaos Zarifis, National Technical University of Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sayan Das, Indian Statistical Institute, Kolkata, India; Zarif Ibragimov, SamSU, Samarkand, Uzbekistan; Harun Immanuel, ITS Surabaya; Titouan Morvan, Lycée Millet, France.