U241. Let $a > b$ be positive real numbers. Prove that

$$c_n = \frac{\sqrt[n+1]{a^{n+1} - b^{n+1}}}{\sqrt[n]{a^n - b^n}}$$

is a decreasing sequence and find its limit.

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Since $\frac{\sqrt[n+1]{a^{n+1} - b^{n+1}}}{\sqrt[n]{a^n - b^n}} = \frac{\sqrt[n+1]{1 - \left(\frac{b}{a}\right)^{n+1}}}{\sqrt[n]{1 - \left(\frac{b}{a}\right)^{n}}}$ and $\frac{b}{a} < 1$, we then find that

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{a^{n+1} - b^{n+1}}}{\sqrt[n]{a^n - b^n}} = \frac{\lim_{n\to\infty} \sqrt[n+1]{1 - \left(\frac{b}{a}\right)^{n+1}}}{\lim_{n\to\infty} \sqrt[n]{1 - \left(\frac{b}{a}\right)^{n}}} = 1.$$

Now, let $t := \frac{a}{b}$. We have $t > 1$ and we note that $c_n > c_{n+1}$ is equivalent with

$$\frac{\sqrt[n+1]{t^{n+1} - 1}}{\sqrt[n]{t^n - 1}} > \frac{\sqrt[n+2]{t^{n+2} - 1}}{\sqrt[n+1]{t^{n+1} - 1}} \text{ i.e. } \left(t^{n+1} - 1\right)^{\frac{2}{n+1}} \geq \left(t^n - 1\right)^{\frac{1}{n}} \left(t^{n+2} - 1\right)^{\frac{1}{n+2}},$$

which rewrites as $\left(t^{n+1} - 1\right)^2 \geq \left(t^n - 1\right)^{\frac{n+1}{n}} \left(t^{n+2} - 1\right)^{\frac{n+1}{n+2}}$.

However, $\left(t^{n+1} - 1\right)^2 \geq \left(t^n - 1\right)\left(t^{n+2} - 1\right)$, since this is just the same thing as saying $t^n + t^{n+2} \geq 2t^{n+1}$, which is obviously true as $(t-1)^2 \geq 0$. Thus, it suffices to prove that

$$\left(t^n - 1\right)\left(t^{n+2} - 1\right) \geq \left(t^n - 1\right)^{\frac{n+1}{n}} \left(t^{n+2} - 1\right)^{\frac{n+1}{n+2}},$$

or equivalently $\left(t^{n+2} - 1\right)^{\frac{1}{n+2}} \geq \left(t^n - 1\right)^{\frac{1}{n}}$. But now, since $\frac{t^{n+1} - 1}{t^n - 1} \geq t$ iff $t \geq 1$, we have that $\left(\frac{t^{n+1} - 1}{t^n - 1}\right)^n \geq t^n > t^n - 1$, which gives $\left(t^{n+1} - 1\right)^n > \left(t^n - 1\right)^{n+1}$, so $\left(t^{n+1} - 1\right)^{\frac{1}{n+1}} > \left(t^n - 1\right)^{\frac{1}{n}}$, and therefore, $\left(t^{n+2} - 1\right)^{\frac{1}{n+2}} \geq \left(t^n - 1\right)^{\frac{1}{n}}$. This completes the proof.

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