U191. For a positive integer n define $a_n = \prod_{k=1}^{n}(1+\frac{1}{2^k})$. Prove that

$$2-\frac{1}{2^n} \le a_n < 3-\frac{1}{2^{n-1}}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

First solution by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

$1+\sum_{k=1}^{n}\frac{1}{2^k} \le \prod_{k=1}^{n}\left(1+\frac{1}{2^k}\right)$, with equallity only for $n=1$. Since $\sum_{k=1}^{n}\frac{1}{2^k} = \frac{1/2-1/2^{n+1}}{1/2} = 1-\frac{1}{2^n}$ the LHS inequality is obtained.

For the RHS inequality, taking logarithms we have

$$\ln a_n = \ln\prod_{k=1}^{n}\left(1+\frac{1}{2^k}\right) = \sum_{k=1}^{n}\ln\left(1+\frac{1}{2^k}\right) < \sum_{k=1}^{n}\frac{1}{2^k} = 1-\frac{1}{2^n}.$$

Therefore $a_n < e^{1-\frac{1}{2^n}}$. Note that this upper bound for a_n is sharper than the proposed in the problem.

Second solution by Arkady Alt, San Jose, USA

1. Right hand link of double inequality (by Math. Induction).

Let $b_n := 2-\frac{1}{2^n}, n\in\mathbb{N}$.We will prove that $\frac{b_{n+1}}{b_n} \le \frac{a_{n+1}}{a_n}, n\in\mathbb{N}$.

$$\frac{b_{n+1}}{b_n} \le \frac{a_{n+1}}{a_n} \iff \frac{2-\frac{1}{2^{n+1}}}{2-\frac{1}{2^n}} \le 1+\frac{1}{2^{n+1}} \iff \frac{\frac{1}{2^{n+1}}}{2-\frac{1}{2^n}} \le \frac{1}{2^{n+1}} \iff 1 \le 2-\frac{1}{2^n} \iff 1 \le 2^n.$$

Note that $2-\frac{1}{2^1} = a_1$.Since $\frac{b_{n+1}}{b_n} \le \frac{a_{n+1}}{a_n}, n\in\mathbb{N}$ then from supposition $a_n \le b_n$, $n\in\mathbb{N}$ follows $b_{n+1} = b_n\cdot\frac{b_{n+1}}{b_n} \le a_n\cdot\frac{a_{n+1}}{a_n} = a_{n+1}$.

2. Left hand link of double inequality.

Proof 1.

For $n=1,2$ inequality $a_n < 3-\frac{1}{2^{n-1}}$ holds (by direct calculation).

Let $n\ge 3$. Since $1+\frac{1}{2^k} < e^{\frac{1}{2^k}}$ then $\prod_{k=1}^{n}\left(1+\frac{1}{2^k}\right) < e^{\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}} = e^{1-\frac{1}{2^n}} < e$.

We have $e < 2.8 < 3-\frac{1}{2^{n-1}}$ for $n\ge 3$.

Proof 2.

By AM-GM inequality we have

$$\prod_{k=1}^{n}\left(1+\frac{1}{2^k}\right) \le \left(\frac{\sum_{k=1}^{n}\left(1+\frac{1}{2^k}\right)}{n}\right)^n = \left(1+\frac{\sum_{k=1}^{n}\frac{1}{2^k}}{n}\right)^n < \left(1+\frac{1}{n}\right)^n < e < 2.8 < 3-\frac{1}{2^{n-1}}$$

for $n\ge 3$.

Also solved by Omran Kouba, Institute for Applied Sciences and Technology, Syria; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Andrea Ligori Università di Roma "Tor Vergata", Italy; Lorenzo Pascali Università di Roma "La Sapienza", Italy.