Undergraduate problems

U181. Consider sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(B_n)_{n\geq 2}$, where $a_0 = b_0 = 1$, $a_{n+1} = a_n + b_n$, $b_{n+1} = (n^2+n+1)a_n + b_n$, $n \geq 1$. Evaluate $\lim_{n\to\infty} B_n$, where

$$B_n = \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}.$$

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Proposition 1.

$$a_n = 2n!, n \geq 1.$$

Proof. By replacing (b_n, b_{n+1}) with $(a_{n+1} - a_n, a_{n+2} - a_{n+1})$ in

$$b_{n+1} = \left(n^2+n+1\right) a_n + b_n$$

we obtain

$$a_{n+2} - a_{n+1} = \left(n^2+n+1\right) a_n + a_{n+1} - a_n \iff a_{n+2} = 2a_{n+1} + \left(n^2+n\right) a_n, n \geq 0.$$

We also have

$$a_1 = a_0 + b_0 = 2, b_1 = \left(0^2+0+1\right) a_0 + b_0 = 2, a_2 = a_1 + b_1 = 4.$$

Since $a_1 = 2\cdot 1!, a_2 = 2\cdot 2!$ and from supposition $a_n = 2n!, a_{n+1} = 2\,(n+1)!$ it follows that

$$\begin{aligned} a_{n+2} &= 2\cdot 2\,(n+1)! + n\,(n+1)\cdot 2n! = 2n!\,(2n+2+n\,(n+1)) \\ &= 2n!\left(n^2+3n+2\right) = 2n!\,(n+1)\,(n+2) \\ &= 2\,(n+2)!. \end{aligned}$$

Then by math induction we have $a_n = 2n!, n \geq 1$.■ Thus,

$$B_n = \frac{(n+1)^2}{\sqrt[n+1]{2\,(n+1)!}} - \frac{n^2}{\sqrt[n]{2n!}}.$$

Proposition 2.

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e.$$

Proof. We have

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}.$$

Since

$$\left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k}\right)^{k+1} \iff \frac{(k+1)^k}{k^k} < e < \frac{(k+1)^{k+1}}{k^{k+1}}, k \in \mathbb{N}$$

then

$$\prod_{k=1}^{n}\frac{(k+1)^k}{k^k}<e^n<\prod_{k=1}^{n}\frac{(k+1)^{k+1}}{k^{k+1}} \iff \prod_{k=1}^{n}\frac{(k+1)^k}{k^{k-1}}\cdot\prod_{k=1}^{n}\frac{1}{k}<e^n<\prod_{k=1}^{n}\frac{(k+1)^{k+1}}{k^k}\cdot\prod_{k=1}^{n}\frac{1}{k}$$

$$\iff \frac{(n+1)^n}{n!}<e^n<\frac{(n+1)^{n+1}}{n!} \iff \left(\frac{n+1}{e}\right)^n<n!<\frac{(n+1)^{n+1}}{e^n}$$

$$\iff \frac{n+1}{e}<\sqrt[n]{n!}<\frac{(n+1)\sqrt[n]{n+1}}{e}$$

$$\iff \frac{ne}{(n+1)\sqrt[n]{n+1}}<\frac{n}{\sqrt[n]{n!}}<\frac{ne}{n+1}.$$

Hence,

$$\lim_{n\to\infty}\frac{ne}{n+1}=\lim_{n\to\infty}\frac{ne}{(n+1)\sqrt[n]{n+1}}=e \implies \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e.\blacksquare$$

Proposition 3.

$$\lim_{n\to\infty}\left(\frac{n^2}{\sqrt[n]{n!}}-\frac{n^2}{\sqrt[n]{2n!}}\right)=e\ln 2.$$

Proof. Since $\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e$ and $\lim_{n\to\infty}n\left(\sqrt[n]{2}-1\right)=\ln 2\lim_{n\to\infty}\frac{n}{\ln 2}\left(e^{\frac{\ln 2}{n}}-1\right)=\ln 2$ then

$$\lim_{n\to\infty}\left(\frac{n^2}{\sqrt[n]{n!}}-\frac{n^2}{\sqrt[n]{2n!}}\right)=\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{2n!}}\left(\sqrt[n]{2}-1\right)=\lim_{n\to\infty}\frac{1}{\sqrt[n]{2}}\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\cdot\lim_{n\to\infty}n\left(\sqrt[n]{2}-1\right)=e\ln 2.\blacksquare$$

Let $\delta_n=\frac{n^2}{\sqrt[n]{2n!}}-\frac{n^2}{\sqrt[n]{n!}}+e\ln 2$ then

$$\lim_{n\to\infty}\delta_n=0$$

and

$$\frac{n^2}{\sqrt[n]{2n!}}=\frac{n^2}{\sqrt[n]{n!}}-e\ln 2+\delta_n.$$

Let $\overline{B}_n=\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}$, then

$$B_n=\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-e\ln 2+\delta_{n+1}-\left(\frac{n^2}{\sqrt[n]{n!}}-e\ln 2+\delta_n\right)=\overline{B}_n+\delta_{n+1}-\delta_n$$

and, therefore,

$$\lim_{n\to\infty}\left(B_n-\overline{B}_n\right)=0.$$

To prove following proposition we need inequality for $n!$ which is more accurate then the one above - which help us find

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}.$$

For our purpose it the following is sufficient.

Lemma.(The proof is in Appendix). *There is positive constant a such that for any $n\in\mathbb{N}$ the following inequality holds*

$$\left(\frac{n}{e}\right)^n\sqrt{an}<n!<\left(\frac{n}{e}\right)^n\sqrt{an}\cdot e^{\frac{1}{12n}}. \tag{1}$$

Proposition 4.

$$\lim_{n\to\infty}(n+1)\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}-\frac{n}{\sqrt[n]{n!}}\right)=0.$$

Proof. Since

$$(1) \iff \frac{n}{e}\sqrt[2n]{an} < \sqrt[n]{n!} < \frac{n}{e}\sqrt[2n]{an}\cdot e^{\frac{1}{12n^2}} \iff \frac{e^{1-\frac{1}{12n^2}}}{\sqrt[2n]{an}} < \frac{n}{\sqrt[n]{n!}} < \frac{e}{\sqrt[2n]{an}}$$

we have

$$\frac{n+1}{\sqrt[n+1]{(n+1)!}} - \frac{n}{\sqrt[n]{n!}} < \frac{e}{\sqrt[2(n+1)]{a(n+1)}} - \frac{e^{1-\frac{1}{12n^2}}}{\sqrt[2n]{an}} = \frac{e}{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}\left(\frac{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}{\sqrt[2(n+1)]{a(n+1)}} - 1\right)$$
$$= \frac{e}{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}\left(e^{\alpha_n} - 1\right),$$

where

$$\alpha_n = \ln\frac{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}{\sqrt[2(n+1)]{a(n+1)}} = \frac{1}{2}\left(\frac{\ln an}{n} - \frac{\ln a(n+1)}{n+1} + \frac{1}{6n^2}\right).$$

Also note that $\frac{n+1}{\sqrt[n+1]{(n+1)!}} > \frac{n}{\sqrt[n]{n!}}, n \in \mathbb{N}$. Indeed,

$$\frac{n+1}{\sqrt[n+1]{(n+1)!}} > \frac{n}{\sqrt[n]{n!}} \iff \frac{(n+1)^{n(n+1)}}{((n+1)!)^n} > \frac{n^{n(n+1)}}{(n!)^{n+1}}$$
$$\iff \frac{(n+1)^{n^2}}{(n!)^n} > \frac{n^{n(n+1)}}{(n!)^{n+1}}$$
$$\iff n! > \frac{n^{n(n+1)}}{(n+1)^{n^2}}$$
$$\iff n! > \left(\frac{n}{\left(1+\frac{1}{n}\right)^n}\right)^n \Longleftarrow n! > \left(\frac{n}{e}\right)^n.$$

Thus,

$$0 < (n+1)\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}} - \frac{n}{\sqrt[n]{n!}}\right) < \frac{(n+1)e}{\sqrt[2n]{an}e^{\frac{1}{12n^2}}}\left(e^{\alpha_n} - 1\right).$$

Since $\lim\limits_{n\to\infty} \frac{e}{\sqrt[2n]{an}e^{\frac{1}{12n^2}}} = e$ and $\lim\limits_{n\to\infty} \alpha_n = 0$ yields $\lim\limits_{n\to\infty} \frac{e^{\alpha_n}-1}{\alpha_n} = 1$ then it suffices to prove that

$$\lim_{n\to\infty}(n+1)\alpha_n = 0.$$

We have

$$\lim_{n\to\infty}(n+1)\alpha_n = \frac{1}{2}\lim_{n\to\infty}(n+1)\left(\frac{\ln an}{n} - \frac{\ln a(n+1)}{n+1} + \frac{1}{6n^2}\right)$$
$$= \frac{1}{2}\lim_{n\to\infty}(n+1)\left(\frac{\ln an}{n} - \frac{\ln an}{n+1} + \frac{\ln an}{n+1} - \frac{\ln a(n+1)}{n+1} + \frac{1}{6n^2}\right)$$
$$= \frac{1}{2}\lim_{n\to\infty}(n+1)\left(\frac{\ln an}{n(n+1)} - \frac{\ln\left(1+\frac{1}{n}\right)}{n+1} + \frac{1}{6n^2}\right)$$
$$= \frac{1}{2}\left(\lim_{n\to\infty}\frac{\ln an}{n} - \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right) + \lim_{n\to\infty}\frac{n+1}{6n^2}\right) = 0.\blacksquare$$

Hence,

$$\lim_{n\to\infty}\overline{B}_n = \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2+n}{\sqrt[n]{n!}} + \frac{n}{\sqrt[n]{n!}}\right) = \lim_{n\to\infty}(n+1)\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}} - \frac{n}{\sqrt[n]{n!}}\right) + \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = e$$

and, therefore,

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty}\left(\left(B_n - \overline{B}_n\right) + \overline{B}_n\right) = \lim_{n\to\infty}\left(\left(B_n - \overline{B}_n\right)\right) + \lim_{n\to\infty} \overline{B}_n = e.$$

Appendix. Proof of the Lemma.

1. First we will prove inequality

$$e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}},$$

for any $n \in \mathbb{N}$.

Note that sequence $\left(\left(1+\frac{2}{n-1}\right)^n\right)_{n\geq 2}$ is decreasing. Indeed,

$$\begin{aligned}\left(1+\frac{2}{n-1}\right)^n > \left(1+\frac{2}{n}\right)^{n+1} &\iff \left(\frac{n+1}{n-1}\right)^n > \left(\frac{n+2}{n}\right)^{n+1}\\ &\iff \left(\frac{n\,(n+1)}{(n-1)\,(n+2)}\right)^n > 1+\frac{2}{n}\\ &\iff \left(1+\frac{2}{(n-1)\,(n+2)}\right)^n > 1+\frac{2}{n}.\end{aligned}$$

Applying the inequality $(1+a)^n \geq 1 + na + \frac{n(n-1)}{2}a^2, a > 0, n \in \mathbb{N}$ to $a = \frac{2}{(n-1)(n+2)}$ yields

$$\begin{aligned}\left(1+\frac{2}{(n-1)\,(n+2)}\right)^n &\geq 1 + \frac{2n}{(n-1)\,(n+2)} + \frac{n\,(n-1)}{2}\cdot\frac{4}{(n-1)^2\,(n+2)^2}\\ &= 1 + \frac{2n}{(n-1)\,(n+2)} + \frac{2n}{(n-1)\,(n+2)^2}\end{aligned}$$

and

$$\begin{aligned}1 + \frac{2n}{(n-1)\,(n+2)} + \frac{2n}{(n-1)\,(n+2)^2} > 1 + \frac{2}{n} &\iff (n+3)\,n^2 > (n-1)\,(n+2)^2\\ &\iff n^3+3n^2 > n^3+3n^2-4\\ &\iff 4 > 0.\blacksquare\end{aligned}$$

Since $\left(1+\frac{2}{n-1}\right)^n > \left(1+\frac{2}{n}\right)^{n+1} > \left(1+\frac{2}{n+1}\right)^{n+2}$ then

$$\left(1+\frac{2}{n-1}\right)^n > \left(1+\frac{2}{n+1}\right)^{n+2}. \tag{2}$$

By replacing n with $2n+1$ in **(2)** yields

$$\left(1+\frac{1}{n}\right)^{2n+1} > \left(1+\frac{1}{(n+1)}\right)^{2n+3} \iff \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} > \left(1+\frac{1}{n+1}\right)^{n+1+\frac{1}{2}}$$

and, since $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} = e$ then

$$e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}},\ n \in \mathbb{N}.$$

2. Consider Taylor representation for the function $\ln \frac{1+x}{1-x}$ on $x \in (0,1)$:

$$\begin{aligned}
\ln \frac{1+x}{1-x} &= \ln(1+x) - \ln(1-x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (-x)^k}{k} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} + \sum_{k=1}^{\infty} \frac{x^k}{k} \\
&= \sum_{k=1}^{\infty} \frac{\left((-1)^{k-1} + 1\right) x^k}{k} \\
&= \sum_{k=1}^{\infty} \frac{2x^{2k-1}}{2k-1}.
\end{aligned}$$

Since $\frac{n+1}{n} = \frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}$ then by replacing x with $\frac{1}{2n+1}$ in $\ln \frac{1+x}{1-x} = \sum\limits_{k=1}^{\infty} \frac{2x^{2k-1}}{2k-1}$ we obtain

$$\begin{aligned}
\ln \frac{n+1}{n} &= \sum_{k=1}^{\infty} \frac{2}{2k-1} \cdot \frac{1}{(2n+1)^{2k-1}} = \frac{2}{2n+1} \left(1 + \sum_{k=2}^{\infty} \frac{1}{(2k-1)(2n+1)^{2(k-1)}}\right) \\
&< \frac{2}{2n+1} \left(1 + \sum_{k=1}^{\infty} \frac{1}{3(2n+1)^{2k}}\right) = \frac{2}{2n+1} \left(1 + \frac{1}{3} \cdot \frac{\frac{1}{(2n+1)^2}}{1 - \frac{1}{(2n+1)^2}}\right) \\
&= \frac{1}{n+\frac{1}{2}} \left(1 + \frac{1}{12n(n+1)}\right) \implies \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}.
\end{aligned}$$

Thus we have the double inequality

$$e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}.$$

Since

$$\begin{aligned}
e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} &\iff e < \frac{(n+1)^{n+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \iff e(n+1) < \frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \\
&\iff \frac{e^{n+1}(n+1)!}{e^n n!} < \frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \\
&\iff \frac{n^{n+\frac{1}{2}}}{e^n n!} < \frac{(n+1)^{(n+1)+\frac{1}{2}}}{e^{n+1}(n+1)!}, n \geq 1
\end{aligned}$$

then the sequence $(a_n)_{n\geq 1}$ where $a_n = \frac{n^{n+\frac{1}{2}}}{e^n n!}$, is increasing. Since

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}} &\iff \frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}} < \frac{(n+1)!}{n!} \cdot e^{n+1-n+\frac{1}{12n}-\frac{1}{12(n+1)}} \\
&\iff \frac{(n+1)^{n+1+\frac{1}{2}}}{(n+1)! e^{n+1-\frac{1}{12(n+1)}}} < \frac{n^{n+\frac{1}{2}}}{n! e^{n-\frac{1}{12n}}}
\end{aligned}$$

then the sequence $(b_n)_{n\geq 1}$, where $b_n = \frac{n^{n+\frac{1}{2}} e^{\frac{1}{12n}}}{n! e^n} = a_n e^{\frac{1}{12n}}$ is decreasing. Since $e^{-1} = a_1 \leq a_n < b_n \leq b_1 = e^{-\frac{11}{12}}$ and $b_n = a_n e^{\frac{1}{12n}}$ then both sequences converge to the same limit. Let $\frac{1}{a} = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$

then $e^{\frac{11}{12}} < a < e$ and

$$a_n < \frac{1}{a} < b_n \iff \frac{1}{b_n} < a < \frac{1}{a_n} \iff \frac{n!e^n}{n^{n+\frac{1}{2}}e^{\frac{1}{12n}}} < a < \frac{e^n n!}{n^{n+\frac{1}{2}}}$$
$$\iff \left(\frac{n}{e}\right)^n \sqrt{an} < n! < \left(\frac{n}{e}\right)^n \sqrt{an} \cdot e^{\frac{1}{12n}}.$$

Remark. Using Vallis' formula we can obtain $a = 2\pi$ but here we don't need that.

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