U178. Let k be a fixed positive integer and let $S_n^{(j)} = \binom{n}{j} + \binom{n}{j+k} + \binom{n}{j+2k} + \cdots,\ j = 0, 1, \ldots, k-1$. Prove that

$$\left(S_n^{(0)} + S_n^{(1)} \cos\frac{2\pi}{k} + \cdots + S_n^{(k-1)} \cos\frac{2(k-1)\pi}{k}\right)^2$$
$$+ \left(S_n^{(1)} \sin\frac{2\pi}{k} + S_n^{(2)} \sin\frac{4\pi}{k} + \cdots + S_n^{(k-1)} \sin\frac{2(k-1)\pi}{k}\right)^2 = \left(2\cos\frac{\pi}{k}\right)^{2n}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Arkady Alt, San Jose, California, USA

Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $D_j = \{j + mk \mid m \in \mathbb{Z}_+ \text{ and } j + mk \le n\}$. Then $S_n^{(j)} = \sum\limits_p \binom{n}{p}$ and $\bigcup\limits_{j=0}^{k-1} D_j = \{0, 1, 2, \ldots, n\}$. Let

$$a = \sum_{j=0}^{k-1} S_n^{(j)} \cos\frac{2j\pi}{k}$$
$$b = \sum_{j=0}^{k-1} S_n^{(j)} \sin\frac{2j\pi}{k}$$
$$\varepsilon = \cos\frac{2\pi}{k} + i\sin\frac{2\pi}{k} \cdot \sum_{j=0}^{k-1} S_n^{(j)} \left(\cos\frac{2\pi}{k} + i\sin\frac{2\pi}{k}\right)^j.$$

Then $\varepsilon^k = 1$ and

$$\begin{aligned}
a + ib &= \sum_{j=0}^{k-1} S_n^{(j)} \cos\frac{2j\pi}{k} + i\sum_{j=0}^{k-1} S_n^{(j)} \cos\frac{2j\pi}{k} \\
&= \sum_{j=0}^{k-1} S_n^{(j)} \left(\cos\frac{2j\pi}{k} + i\sin\frac{2j\pi}{k}\right) \\
&= \sum_{j=0}^{k-1} S_n^{(j)} \varepsilon^j = \sum_{j=0}^{k-1} \sum_{p \in D_j} \binom{n}{p} \varepsilon^p \\
&= \sum_{p \in \cup D_j} \binom{n}{p} \varepsilon^p = \sum_{p=1}^{n} \binom{n}{p} \varepsilon^p = (1+\varepsilon)^n \\
&= \left(1 + \cos\frac{2\pi}{k} + i\sin\frac{2\pi}{k}\right)^n = \left(2\cos\frac{\pi}{k}\left(\cos\frac{\pi}{k} + i\sin\frac{\pi}{k}\right)\right)^n \\
&= \left(2\cos\frac{\pi}{k}\right)^n \left(\cos\frac{\pi}{k} + i\sin\frac{\pi}{k}\right)^n.
\end{aligned}$$

Hence,

$$\begin{aligned}|a+ib| &= \left|\left(2\cos\frac{\pi}{k}\right)^n\left(\cos\frac{\pi}{k}+i\sin\frac{\pi}{k}\right)^n\right| = \left|\left(2\cos\frac{\pi}{k}\right)^n\right|\left|\left(\cos\frac{\pi}{k}+i\sin\frac{\pi}{k}\right)^n\right| \\ &= \left|\left(2\cos\frac{\pi}{k}\right)\right|^n\left|\left(\cos\frac{\pi}{k}+i\sin\frac{\pi}{k}\right)\right|^n = \left|\left(2\cos\frac{\pi}{k}\right)\right|^n.\end{aligned}$$

Therefore, $a^2 + b^2 = \left(2\cos\frac{\pi}{k}\right)^{2n}$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Campos Salas, Costa Rica.