U171. Let A be a matrix of order n such that $A^{10} = O_n$. Prove that

$$\frac{1}{4}A^4 + \frac{1}{2}A^3 + \frac{1}{2}A^2 + A + I_n$$

is invertible.

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Solution by Arkady Alt, San Jose, California, USA

Let $f(A) = -\frac{1}{4}A^3 - \frac{1}{2}A^3 - \frac{1}{2}A - I_n$ and $B = Af(A)$. Since for any two polynomial $P(x)$ and $Q(x)$ and any matrix A holds $P(A)Q(A) = Q(A)P(A)$ then $B^{10} = (Af(A))^{10} = A^{10}f(A)^{10} = 0_n$.Hence, $\left(\frac{1}{4}A^4 + \frac{1}{2}A^3 + \frac{1}{2}A^2 + A + I_n\right)\left(I_n + B + B^2 + \cdots + B^9\right) = (I_n - B)\left(I_n + B + B^2 + \cdots + B^9\right) = I_n - B^{10} = I_n$.Thus, $I_n + B + B^2 + \cdots + B^9$ is inverse matrix for $\frac{1}{4}A^4 + \frac{1}{2}A^3 + \frac{1}{2}A^2 + A + I_n$.

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