U167. Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function such that $f(1) = 0$. Prove that

$$\left|\int_0^1 xf(x)dx\right| \leq \frac{1}{6}\max_{x\in[0,1]}|f'(x)|.$$

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Solution by Arkady Alt, San Jose, California, USA

Using integration by parts we obtain $\int_0^1 xf(x)\,dx = \left(\frac{x^2}{2}\cdot f(x)\right)_0^1 - \int_0^1 \frac{x^2f'(x)}{2}dx =$ $-\frac{1}{2}\int_0^1 x^2f'(x)\,dx$. Since by condition $f(x)$ is continuously differentiable then $M := \max\limits_{x\in[0,1]}|f'(x)|$ and, therefore, $\left|\int_0^1 xf(x)\,dx\right| = \left|-\frac{1}{2}\int_0^1 x^2f'(x)\,dx\right| =$ $\frac{1}{2}\left|\int_0^1 x^2f'(x)\,dx\right| \leq \frac{1}{2}\int_0^1 x^2|f'(x)|\,dx \leq \frac{M}{2}\int_0^1 x^2dx = \frac{1}{6}\max\limits_{x\in[0,1]}|f'(x)|$..

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