U159. Let x and y be positive real numbers. Prove that

$$x^y y^x \le \left(\frac{x+y}{2}\right)^{x+y}.$$

Proposed by Samuel G. Moreno, Universidad de Jaén, Spain

First solution by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

From weighted AM–GM inequality we conclude that

$$x^y y^x \le \left(\frac{xy+yx}{x+y}\right)^{x+y} = \left(\frac{2xy}{x+y}\right)^{x+y} \le \left(\frac{x+y}{2}\right)^{x+y},$$

where the last inequality again follows from AM–GM

$$\frac{2xy}{x+y} \le \frac{x+y}{2} \Leftrightarrow \sqrt{xy} \le \frac{x+y}{2}.$$

Second solution by Arkady Alt, San Jose, California, USA

Let us take a more indepth look at this inequality. Firstly, we will prove the stronger inequality

$$x^y y^x \le (xy)^{\frac{x+y}{2}}.$$

Indeed,

$$x^y y^x \le (xy)^{\frac{x+y}{2}} \iff x^{2y} y^{2x} \le x^{x+y} y^{x+y} \iff x^y y^x \le x^x y^y \iff 1 \le \left(\frac{x}{y}\right)^{x-y}.$$

Since

$$(xy)^{\frac{1}{2}} \le \frac{x+y}{2} \iff (xy)^{\frac{x+y}{2}} \le \left(\frac{x+y}{2}\right)^{x+y}$$

we obtain $x^y y^x \le \left(\frac{x+y}{2}\right)^{x+y}$.

Secondly, by weighted AM–GM,

$$x^{\frac{y}{x+y}} y^{\frac{x}{x+y}} \le x \cdot \frac{y}{x+y} + y \cdot \frac{x}{x+y} = \frac{2xy}{x+y}$$

then original inequality immediately follows from the obtained stronger inequality

$$x^y y^x \le \left(\frac{2xy}{x+y}\right)^{x+y}$$

and the inequality $\frac{2xy}{x+y} \le \frac{x+y}{2}$. Thus we obtained the chain of inequalities

$$x^y y^x \le \left(\frac{2xy}{x+y}\right)^{x+y} \le (xy)^{\frac{x+y}{2}} \le \left(\frac{x+y}{2}\right)^{x+y}.$$

Remark. Normalization of the inequality $x^y y^x \le \left(\dfrac{2xy}{x+y}\right)^{x+y}$ by the condition $x+y=1$ yields

$$x^y y^x \le 2xy \iff \frac{1}{2} \le x^x y^y \iff \frac{1}{2} \le x^x (1-x)^{1-x}, x \in (0,1)$$

which can proved by employing calculus. Let

$$h(x) = x \ln x + (1-x) \ln (1-x)$$

then

$$h'(x) = \ln \frac{x}{1-x}$$

and

$$h'(x) < 0 \iff x \in \left(0, \frac{1}{2}\right), h'(x) > 0 \iff x \in \left(\frac{1}{2}, 1\right), h'(x) = 0 \iff x = \frac{1}{2}.$$

Thus

$$\min_{x \in (0,1)} h(x) = h\left(\frac{1}{2}\right),$$

i.e.,

$$h(x) \ge h\left(\frac{1}{2}\right) = \ln \frac{1}{2} \iff \frac{1}{2} \le x^x (1-x)^{1-x}.$$

Also solved by Ajat Adriansyah, Universitas Indonesia, Indonesia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; G. C. Greubel, Newport News, USA; Magkos Athanasios, Kozani, Greece; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Samuel G. Moreno, Universidad de Jaen, Spain; Samin Riasat, Notre Dame College, Dhaka, Bangladesh; Sayan Mukherjee, Kolkata, India.