Undergraduate problems

U151. Let n be a positive integer and let

$$f(x) = x^{n+8} - 10x^{n+6} + 2x^{n+4} - 10x^{n+2} + x^n + x^3 - 10x + 1.$$

Evaluate $f(\sqrt{2} + \sqrt{3})$.

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First solution by John Mangual, UC Santa Barbara, USA

Let $x = \sqrt{2} + \sqrt{3}$. Then $x^2 = 5 + 2\sqrt{6}$ and more manipulation gives $x^4 - 10x^2 + 1 = 0$. The polynomial to be evaluated can be written

$$(x^n + x^{n+4})(x^4 - 10x^2 + 1) + x^3 - 10x + 1.$$

The first term vanishes and we evaluate the remaining trinomial. Using the binomial theorem $(\sqrt{2} + \sqrt{3})^3 = 11\sqrt{2} + 9\sqrt{3}$. Finally

$$(\sqrt{2} + \sqrt{3})^3 - 10(\sqrt{2} + \sqrt{3}) + 1 = (11\sqrt{2} + 9\sqrt{3}) - 10(\sqrt{2} + \sqrt{3}) + 1 = \sqrt{2} - \sqrt{3} + 1.$$

Second solution by Arkady Alt, San Jose, California, USA

Note that $\sqrt{2} + \sqrt{3}$ is a root of the polynomial

$$x^4 - 10x^2 + 1 = \left(x - \sqrt{2} - \sqrt{3}\right)\left(x + \sqrt{2} - \sqrt{3}\right)\left(x + \sqrt{2} + \sqrt{3}\right)\left(x - \sqrt{2} + \sqrt{3}\right),$$

Because

$$\begin{aligned} x^{n+8} - 10x^{n+6} + 2x^{n+4} - 10x^{n+2} + x^n + x^3 - 10x + 1 &= \left(x^4 - 10x^2 + 1\right)\left(x^{n+4} + x^n\right) + x^3 - 10x + 1 \\ &= \left(x^4 - 10x^2 + 1\right)\left(x^{n+4} + x^n + \frac{1}{x}\right) + 1 - \frac{1}{x} \end{aligned}$$

then

$$f\left(\sqrt{2} + \sqrt{3}\right) = 1 - \frac{1}{\sqrt{3} + \sqrt{2}} = 1 + \sqrt{2} - \sqrt{3}.$$

Also solved by Américo Tavares, Queluz, Portugal; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Lopez Aguayo, Puebla, Mexico; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Emanuele Natale, Università di Roma "Tor Vergata", Roma, Italy.