U128. Let f be a twice differentiable continuous real-valued function defined on $[0,1]$ such that $f(0)=f(1)=f'(1)=0$ and $f'(0)=1$. Prove that

$$\int_0^1 \left(f''(x)\right)^2 dx \geq 4.$$

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By Cauchy Inequality we have

$$\int_0^1 (3x-2)^2\, dx \cdot \int_0^1 \left(f''(x)\right)^2 dx \geq \left(\int_0^1 (3x-2)\, f''(x)\, dx\right)^2$$

and since

$$\int_0^1 (3x-2)^2\, dx = \left(\frac{(3x-2)^3}{9}\right)_0^1 = \frac{1}{9}+\frac{8}{9}=1,$$

$$\int_0^1 (3x-2)\, f''(x)\, dx = \left[\begin{array}{c} u'=f''(x)\,; u=f'(x) \\ v=3x-2; v'=3 \end{array}\right] = \left((3x-2)\, f'(x)\right)_0^1 - 3\int_0^1 f'(x)\, dx =$$

$$\left(1\cdot f'(1)-(-2)\cdot f'(0)\right)-3\left(f(1)-f(0)\right)=2$$

then $\int_0^1 \left(f''(x)\right)^2 dx \geq 4$.Equality occurs if $f(x)=x(x-1)^2$.

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