U124. Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers such that $\arctan x_n + nx_n = 1$ for all positive integers n. Evaluate $\lim_{n\to\infty} n\ln(2-nx_n)$.

Proposed by Duong Viet Thong, Nam Dinh University of Technology and Education, Vietnam

First solution by Arkady Alt, San Jose, California, USA

First note that for any positive real x holds inequality $\arctan x < x$.(This immmediatelly

follows from inequality $x < \tan x, x \in \left(0, \frac{\pi}{2}\right)$).

Since function $f(x) := \arctan x + nx$ is odd in $\mathbb{R}$, then from $f(x) = 1$ follows $x > 0$.

Thus, all terms of sequence $\{x_n\}_{n\geq 1}$ determined by equation $\arctan x_n + nx_n = 1$ should

be positive and for any natural n holds inequality

$$\arctan x_n < x_n \iff 1 - nx_n < x_n \iff \frac{1}{n+1} < x_n.$$

From the other hand since $x_n > 0$ then $\arctan x_n > 0 \implies 1 - nx_n > 0 \iff x_n < \frac{1}{n}$.

Thus, $\frac{1}{n+1} < x_n < \frac{1}{n}, n \in \mathbb{N}$ and, therefore, $\lim\limits_{n\to\infty} nx_n = 1$, $\lim\limits_{n\to\infty} \arctan x_n = 0$.

Moreover, $\lim\limits_{n\to\infty} n\arctan x_n = \lim\limits_{n\to\infty}\left(\frac{\arctan x_n}{x_n}\cdot nx_n\right) = \lim\limits_{n\to\infty}\frac{\arctan x_n}{x_n}\cdot \lim\limits_{n\to\infty} nx_n = 1\cdot 1 = 1$.

Using this we obtain $\lim\limits_{n\to\infty} n\ln(2-nx_n) = \lim\limits_{n\to\infty} n\ln(1+\arctan x_n) =$
$\lim\limits_{n\to\infty}\left(\frac{\ln(1+\arctan x_n)}{\arctan x_n}\cdot n\arctan x_n\right) = \lim\limits_{n\to\infty}\frac{\ln(1+\arctan x_n)}{\arctan x_n}\cdot \lim\limits_{n\to\infty} n\arctan x_n = 1\cdot 1 = 1$.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $f(x) = \arctan x$ and $g(x) = \ln(1+x)$. Since $1 - x^2 < f'(x) = \frac{1}{1+x^2} < 1$ and $1 - x < g'(x) = \frac{1}{1+x} < 1$ for $x > 0$, then $x - \frac{x^3}{3} < \arctan x < x$ and $x - \frac{x^2}{2} < \ln(1+x) < x$ for $x > 0$.

If $x_n < 0$, then $-\frac{\pi}{2} < \arctan x_n < 0$, and $\arctan x_n + nx_n < 0$, absurd, hence $x_n > 0$ and $\arctan x_n > 0$, or $nx_n = 1 - \arctan x_n < 1$. Therefore, $x_n - \frac{x_n^3}{3} <$