U117. Let n be an integer greater than 1 and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 + x_2 + \cdots + x_n = n$. Prove that

$$\sum_{k=1}^{n} \frac{x_k}{n^2 - n + 1 - nx_k + (n-1)x_k^2} \le \frac{1}{n-1}$$

and find all equality cases.

Proposed by Iurie Boreico, Harvard University, USA

Solution by Arkady Alt, San Jose, California, USA

Lemma.

For any $a, b, c > 0$ such that $c \ge b,\ b^2 < 4ac,$ and any $t \ge -1$ holds inequality

$$\frac{c^2(t+1)}{at^2 + bt + c} \le (c-b)\,t + c$$

with equality condition $t = 0$.

Proof.

Note that $at^2 + bt + c > 0$ for any t and in particular $a + c - b > 0,$ since $b^2 < 4ac$ and $a > 0$.Then we have

$$\frac{c^2(t+1)}{at^2 + bt + c} \le (c-b)\,t + c \iff 0 \le ((c-b)\,t + c)\left(at^2 + bt + c\right) - c^2(t+1) \iff$$

$t^2\left(a(c-b)\,t + bc + ca - b^2\right) \ge 0$ where latter inequality holds because

$a(c-b)\,t + bc + ca - b^2 = a(c-b)(t+1) + bc + ab - b^2 =$

$a(c-b)(t+1) + b(a + c - b)$ and $c - b \ge 0, t + 1 \ge 0, a + c - b > 0.$

Using the Lemma for $a = n-1, b = n-2, c = n^2 - n$ for which obviously holds $a, b, c > 0$ and $c \ge b,\ b^2 < 4ac$ gives us the following inequality

$$(1) \qquad \frac{t+1}{(n-1)\,t^2 + (n-2)\,t + n^2 - n} \le \frac{\left(n^2 - 2n + 2\right)t + n(n-1)}{n^2(n-1)^2}.$$

Substitution $t = x - 1$ in (1) yields

$$(2) \qquad \frac{x}{n^2 - n + 1 - nx + (n-1)\,x^2} \le \frac{\left(n^2 - 2n + 2\right)x + n - 2}{n^2(n-1)^2}$$

which becomes equality if and only if $x = 1$.

Using (2) we obtain

$$\sum_{k=1}^{n} \frac{x_k}{n^2 - n + 1 - nx_k + (n-1)\, x_k^2} \leq \sum_{k=1}^{n} \frac{\left(n^2 - 2n + 2\right) x_k + n - 2}{n^2\,(n-1)^2} =$$

$$\frac{\left(n^2 - 2n + 2\right) \sum_{k=1}^{n} x_k + (n-2)\, n}{n^2\,(n-1)^2} = \frac{\left(n^2 - 2n + 2\right) n + (n-2)\, n}{n^2\,(n-1)^2} = \frac{1}{n-1}.$$

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain