U113. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that f is a periodic function but it does not have a least period.

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First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Not having a least period means that for all $r > 0$ there exists $T_r < r$ such that $f(x + T_r) = f(x)$ for all $x \in \mathbf{R}$. Let's suppose that f is not constant. This means that there exists two points, $x_1 > x_0$ such that $f(x_1) > f(x_0)$. Without loss of generality we may suppose $f(x_0) = 0$ and $f(x_1) > 0$. Let d be $x_1 - x_0$. The continuity of f implies that in an open neighborhood of x_1, say (a, b), $f(x) > f(x_1)/2$ holds. Starting by x_0 we move toward right doing jumps of lengths $T_{d/2} < d/2$. The value assumed by f after every jump is 0 by periodicity but after a while, inevitably we enter (a, b) and we fall in contradiction because f should assume the value 0 and a value greater than $f(x_1)/2$.

Second solution by Arkady Alt, San Jose, California, USA

Let Ω_+ be set of all positive periods of function f and let $\omega_* = \inf \Omega_+$.

Since by definition "least period" mean least positive period then by problem's condition $\omega_* \notin \Omega_+$.Consider two cases:

1. $\omega_* = 0$. Then for any natural n there is $\omega_n \in \Omega_+$ such that $0 < \omega_n < \dfrac{1}{n}$.

Let x by any real number and let $x_n := x - \omega_n \left\lfloor \dfrac{x}{\omega_n} \right\rfloor$ then $0 \le x < \omega_n$ and, therefore, $\lim_{n\to\infty} x_n = 0$. Since $f(x) = f\left(x_n + \omega_n \left\lfloor \dfrac{x}{\omega_n} \right\rfloor\right) = f(x_n)$ and f is continuous

function then $f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(0)$.

2. $\omega_* > 0$. Then for any natural n there is $\omega_n \in \Omega_+$ such that $\omega_* < \omega_n < \omega_* + \dfrac{1}{n}$.

Let x be any real number. Since $\lim_{n\to\infty} \omega_n = \omega_*$ and f is continuous function then

$f\ (x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} f(x + \omega_n) = f\left(\lim_{n\to\infty} (x + \omega_n)\right) = f(\omega_*)$.

Thus we obtain that ω_* is positive period of f , that is contradiction with $\omega_* \notin \Omega_+$

and this prove that only constant functions is solution of problem.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain