U105. Find $\min\left(\frac{\operatorname{Im} z^5}{\operatorname{Im}^5 z}\right)$ over all z in $\mathbb{C}\backslash\mathbb{R}$.

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Using polar form $z = r(\cos\varphi + i\sin\varphi)$, with $r > 0$ and $\varphi \neq k\pi, k \in \mathbb{Z}$ we obtain

$$\frac{Imz^5}{Im^5 z} = \frac{\sin 5\varphi}{\sin^5\varphi} = \frac{16\sin^5\varphi - 20\sin^3\varphi + 5\sin\varphi}{\sin^5\varphi} = \frac{5}{\sin^4\varphi} - \frac{20}{\sin^2\varphi} + 16 =$$

$5\left(\frac{1}{\sin^2\varphi} - 2\right)^2 - 4 \geq -4$ and, since lower bound -4 for $\frac{Imz^5}{Im^5 z}$ can be attained if and only if

$\sin^2\varphi = \frac{1}{2} \iff \varphi = \frac{(2n+1)\pi}{4}, n \in \mathbb{Z}$, then $\min\limits_{z\ i\in\ \mathbb{C}\setminus\mathbb{R}}\left(\frac{Imz^5}{Im^5 z}\right) = -4.$

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