S98. Let n be a positve integer. Prove that $\prod_{d|n} \frac{\phi(d)}{d} \leq \left(\frac{\phi(n)}{n}\right)^{\frac{\tau(n)}{2}}$, where $\tau(n)$ is the number of divisors of n and $\phi(n)$ is Euler's totient function.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Arkady Alt, San Jose, California, USA

Let $P(n) := \prod_{d|n} \dfrac{\varphi(d)}{d}$ and $F(n) := P(n)^{\frac{1}{\tau(n)}}$.Then, since $\varphi(n)$ is a multiplicative function, for any two relatively prime positive integers n and m holds

$$\begin{aligned} P(nm) &= \prod_{d|nm} \frac{\varphi(d)}{d} = \prod_{s|n} \prod_{t|m} \frac{\varphi(st)}{st} \\ &= \prod_{s|n} \prod_{t|m} \left(\frac{\varphi(s)}{s} \cdot \frac{\varphi(t)}{t} \right) \\ &= \prod_{s|n} \left(\left(\frac{\varphi(s)}{s} \right)^{\tau(m)} \cdot P(m) \right) \\ &= P(m)^{\tau(n)} \cdot \left(\prod_{s|n} \left(\frac{\varphi(s)}{s} \right) \right)^{\tau(m)} \\ &= P(m)^{\tau(n)} \cdot P(n)^{\tau(m)} \end{aligned}$$

and, therefore, due to multiplicativity of $\tau(n)$ we obtain

$$\begin{aligned} P(nm)^{\frac{1}{\tau(nm)}} &= \left(P(n)^{\tau(m)} \cdot P(m)^{\tau(n)} \right)^{\frac{1}{\tau(n)\tau(m)}} \\ &= P(n)^{\frac{1}{\tau(n)}} \cdot P(m)^{\frac{1}{\tau(m)}} \\ &\iff F(nm) = F(n) F(m). \end{aligned}$$

Since

$$\prod_{d|n} \frac{\varphi(d)}{d} \leq \left(\frac{\varphi(n)}{n} \right)^{\frac{\tau(n)}{2}}$$

$$\iff P(n)^{\frac{1}{\tau(n)}} \leq \sqrt{\frac{\varphi(n)}{n}} \iff F(n) \leq \sqrt{\frac{\varphi(n)}{n}}$$

then, using multiplicativity of $F(n)$ and $\varphi(n)$, suffices to prove latter inequality for any $n = q^k$, where q is prime and k is natural number.

Since $\dfrac{\varphi\left(q^i\right)}{q^i}=\dfrac{q^i-q^{i-1}}{q^i}=1-\dfrac{1}{q}$, for $i=1,2,...,k$, $\tau\left(q^k\right)=k+1$, $P\left(q^k\right)=\prod_{d|q^k,d>1}\left(1-\dfrac{1}{q}\right)=\prod_{i=1}^k\left(1-\dfrac{1}{q}\right)=\left(1-\dfrac{1}{q}\right)^k$ then

$$\begin{aligned}
F\left(q^k\right) &\leq \sqrt{\frac{\varphi\left(q^k\right)}{q^k}} \\
&\iff \left(1-\frac{1}{q}\right)^{\frac{k}{k+1}} \leq \left(1-\frac{1}{q}\right)^{\frac{1}{2}} \\
&\iff \left(1-\frac{1}{q}\right)^{\frac{k-1}{2(k+1)}} \leq 1 \iff \left(1-\frac{1}{q}\right)^{k-1} \leq 1.
\end{aligned}$$

Second solution by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

To verify this we only need to note that the number of prime divisors of ab is always less than or equal to the number of primes dividing a plus the number of primes dividing b, with equality iff a and b have no common prime divisor. Therefore the inequality follows, since ϕ contains each prime divisor exactly once.

Now using the inequality above we deduce that

$$\begin{aligned}
\prod_{d|n}\frac{\phi(d)}{d} &= \sqrt{\prod_{d|n}\frac{\phi(d)}{d}\cdot\frac{\phi(n/d)}{n/d}} \\
&\geq \sqrt{\prod_{d|n}\frac{\phi(n)}{n}} \\
&= \sqrt{\left(\frac{\phi(n)}{n}\right)^{\tau(n)}}
\end{aligned}$$

which was what we wanted. Equality holds iff n is square-free i.e. of the form $p_1p_2\cdots p_k$, where the p_i are distinct prime numbers.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Michel Bataille, France.