S88. Let a, b, c, d be non-negative real numbers. Prove that

$$a^2 + b^2 + c^2 + d^2 + 1 + abcd \geq ab + bc + cd + da + ac + bd.$$

Proposed by Alex Anderson, New Trier Township High School, Winnetka, USA

First solution by Brian Bradie, Christopher Newport University, USA

Let w, x, y, z be non-negative real numbers. By Turkevici's inequality

$$w^4 + x^4 + y^4 + z^4 + 2wxyz \geq w^2x^2 + x^2y^2 + y^2z^2 + z^2w^2 + w^2y^2 + x^2z^2, \quad (3)$$

with equality if and only if $w = x = y = z$ or one variable equal to zero and the other three equal to one another. Because w, x, y, z are non-negative real numbers, we can define the non-negative real numbers a, b, c, d according to

$$a = \sqrt{w}, \quad b = \sqrt{x}, \quad c = \sqrt{y}, \quad d = \sqrt{z}.$$

Substituting into (1) yields

$$a^2 + b^2 + c^2 + d^2 + 2\sqrt{abcd} \geq ab + bc + cd + da + ac + bd, \quad (4)$$

with equality if and only if $a = b = c = d$ or one variable equal to zero and the other three equal to one another. Moreover,

$$abcd - 2\sqrt{abcd} + 1 = \left(\sqrt{abcd} - 1\right)^2 \geq 0 \quad (5)$$

with equality if and only if $abcd = 1$. Adding (2) and (3) yields

$$a^2 + b^2 + c^2 + d^2 + 1 + abcd \geq ab + bc + cd + da + ac + bd$$

as desired, with equality if and only if $a = b = c = d$ and $abcd = 1$; that is, if and only if $a = b = c = d = 1$.

An alternate proof proceeds as follows. Because the inequality is symmetric in a, b, c, d, we may assume without loss of generality that $a \geq b \geq c \geq d$. Now

$$\begin{aligned} & a^2 + b^2 + c^2 + d^2 + 1 + abcd - ab - bc - cd - da - ac - bd \\ &= (\sqrt{ab} + \sqrt{cd} - c - d)^2 + 2\sqrt{cd}(\sqrt{c} - \sqrt{d})^2 + \\ &\qquad (\sqrt{a} - \sqrt{b})^2((\sqrt{a} + \sqrt{b})^2 - (c + d)) + (\sqrt{abcd} - 1)^2 \\ &\geq 0, \end{aligned}$$

with equality if and only if $a = b = c = d$ and $abcd = 1$; that is, if and only if $a = b = c = d = 1$.

Second solution by Arkady Alt, San Jose, California, USA

By the AM-GM inequality $1 + abcd \geq 2\sqrt{abcd}$ then it sufficies to prove

$$a^2 + b^2 + c^2 + d^2 + 2\sqrt{abcd} \geq ab + ac + ad + bc + bd + cd. \quad (1)$$

Due to homogeneity we assume that $abcd = 1$. Then inequality (1) becomes

$$a^2 + b^2 + c^2 + d^2 + 2 \geq ab + ac + ad + bc + bd + cd \quad (2)$$

Also, due to the symmetry, we may assume that $d = \min\{a, b, c, d\}$.

Lemma.(Sharp Quadratic Mix Inequality)

For any non-negative a, b, c holds inequality

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \sqrt[3]{abc}\left(a + b + c - 3\sqrt[3]{abc}\right). \quad (3)$$

Proof.

Due to homogeneity assume that $a + b + c = 1$ then, denoting

$p := ab + bc + ca, q := abc,$ obtain (3) $\iff 1 - 3p \geq \sqrt[3]{q}\left(1 - 3\sqrt[3]{q}\right)$.

Since $p \leq \dfrac{1 + 9q}{4} \iff 1 \geq 4p - 9q$ (Schur Inequality $\sum_{cyc} a(a - b)(a - c) \geq 0$ in

p-q notation) and $q = abc \leq \left(\dfrac{a + b + c}{3}\right)^3 = \dfrac{1}{27}$ suffices to prove

$1 - \dfrac{3(1 + 9q)}{4} \geq \sqrt[3]{q}\left(1 - 3\sqrt[3]{q}\right) \iff 1 - 27q \geq 4\sqrt[3]{q}\left(1 - 3\sqrt[3]{q}\right)$ for any $q \in \left[0, \dfrac{1}{27}\right]$.

We have $1 - 27q - 4\sqrt[3]{q} + 12\sqrt[3]{q^2} = \left(1 - 3\sqrt[3]{q}\right)\left(1 + \sqrt[3]{q} + \sqrt[3]{q^2}\right) - 4\sqrt[3]{q}\left(1 - 3\sqrt[3]{q}\right) =$

$\left(1 - 3\sqrt[3]{q}\right)\left(\left(1 - 3\sqrt[3]{q} + \sqrt[3]{q^2}\right)\right) = \left(1 - 3\sqrt[3]{q}\right)^2 + \sqrt[3]{q^2}\left(1 - 3\sqrt[3]{q}\right) \geq 0. \blacksquare$

Since $d = \min\{a, b, c, d\}$ then $abc \geq d^3 \iff \sqrt[3]{abc} \geq d$ and (3) $\implies$

(4) $\quad a^2 + b^2 + c^2 - ab - bc - ca \geq d\left(a + b + c - 3\sqrt[3]{abc}\right)$.

Since (2) $\iff a^2 + b^2 + c^2 - ab - bc - ca \geq d(a + b + c) - d^2 - 2$ then, due to inequality (4), suffices to prove

$d\left(a + b + c - 3\sqrt[3]{abc}\right) \geq d(a + b + c) - d^2 - 2 \iff d^2 + 2 \geq 3d\sqrt[3]{abc}.$

Since $abc = \dfrac{1}{d}$ we have

$$d^2 + 2 - 3d\sqrt[3]{abc} = d^2 + 2 - 3\sqrt[3]{d^2} = \left(d^{\frac{2}{3}} - 1\right)^2\left(d^{\frac{2}{3}} + 1\right) \geq 0.$$

Comment.

Up to notation inequality (2) is Turkevici's Inequality.

Indeed, substitution in (2) $a = x^2, b = y^2, c = z^2, d = t^2,$ where $x, y, z, t \geq 0$ gives us nequality

$x^4 + y^4 + z^4 + t^4 + 2xyzt \geq x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2 + x^2z^2 + y^2t^2.$

Third solution by Ganesh Ajjanagadde, Mysore, India

We are asked to prove that for all $a, b, c, d \geq 0$ the following inequality holds:

$$a^2 + b^2 + c^2 + d^2 + 1 + abcd \geq ab + bc + cd + da + ac + bd \qquad (6)$$

By Turkevici's inequality, we know that

$w^4 + x^4 + y^4 + z^4 + 2wxyz \geq w^2x^2 + w^2y^2 + w^2z^2 + x^2y^2 + x^2z^2 + y^2z^2.$

Substituting $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$ for w, x, y, z respectively in this inequality, we get

$a^2 + b^2 + c^2 + d^2 + 2\sqrt{abcd} \geq ab + ac + ad + bc + bd + cd.$

Thus in order to prove the given inequality, it suffices to show that

$a^2 + b^2 + c^2 + d^2 + 1 + abcd \geq a^2 + b^2 + c^2 + d^2 + 2\sqrt{abcd}$, or, $1 + abcd \geq 2\sqrt{abcd}$, which is clearly true by the AM-GM inequality.

Fourth solution by Perfetti Paolo, Dipartimento di Matematica Tor Vergata Roma, Italy

Proof j1) If $d = 0$ the inequality becomes

$$a^2 + b^2 + c^2 - ab - bc - ac + 1 \geq 0$$

which holds true since $(x^2 + y^2)/2 \geq |xy|$. By continuity the inequality is true also if $d \neq 0$ but sufficiently small, say $0 < d \leq \delta$, because

$$a^2 + b^2 + c^2 - ab - bc - ac \geq 0$$

j2) If $a \to +\infty$ and $b, c, d \leq R$ the inequality (*) holds true because the l.h.s. goes to infinity quadratically while the r.h.s. at most linearly.

j3) By

$$\frac{a^2 + b^2}{2} + c^2 + d^2 + 1 + abcd \geq bc + cd + da + ac + bd$$

it follows that if $a \to +\infty$, $b \to +\infty$ and $c, d \leq R$, (*) is true. $(x^2 + y^2)/2 \geq |xy|$ has been used again