Senior problems

S67. Let ABC be a triangle. Prove that

$$\cos^3 A + \cos^3 B + \cos^3 C + 5\cos A\cos B\cos C \le 1.$$

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First solution by Son Hong Ta, Hanoi, Vietnam

Using the equality

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$$

the initial inequality becomes equivalent to

$$\sum \cos^3 A + 3\prod \cos A \le \sum \cos^2 A,$$

or

$$3\prod \cos A \le \sum \cos^2 A\,(1-\cos A)$$

Now, by the AM-GM inequality, we have

$$\sum \cos^2 A\,(1-\cos A) \ge 3\sqrt[3]{\prod \cos^2 A \cdot \prod (1-\cos A)}$$

Thus, it suffices to prove that

$$\prod \cos A \le \prod (1-\cos A)\,.$$

When triangle ABC is obtuse, the above inequality is clearly true. So we will consider the case it is acute. We have

$$\begin{aligned}
\prod \cos A \le \prod (1-\cos A)& \\
&\iff \prod \cos A\,(1+\cos A) \le \prod \left(1-\cos^2 A\right) \\
&\iff 8\prod \cos A \cdot \prod \cos^2 \frac{A}{2} \le \prod \sin^2 A \\
&\iff \frac{\prod \cos^2 \frac{A}{2}}{\prod \sin \frac{A}{2}\cos \frac{A}{2}} \le \frac{\prod \sin A}{\prod \cos A} \\
&\iff \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \le \tan A\tan B\tan C \\
&\iff \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \le \tan A + \tan B + \tan C
\end{aligned}$$

Indeed, we have

$$\sum \tan A = \sum \frac{\tan B + \tan C}{2} \geq \sum \tan \frac{B+C}{2} = \sum \cot \frac{A}{2},$$

and the equality holds if and only if triangle ABC is equilateral.

Second solution by Arkady Alt, San Jose, California, USA

Since $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$ we will prove that

$$\sum_{cyc} \cos^2 A\,(1-\cos A) \geq 3\cos A \cos B \cos C.$$

By the AM-GM Inequality we have

$$\sum_{cyc} \cos^2 A\,(1-\cos A) \geq 3\sqrt[3]{\prod_{cyc} \cos^2 A\,(1-\cos A)},$$

then it suffices to prove

$$3\sqrt[3]{\prod_{cyc} \cos^2 A\,(1-\cos A)} \geq 3\cos A \cos B \cos C$$

$$\iff \prod_{cyc} (1-\cos A) \geq \cos A \cos B \cos C$$

$$\iff \cos A \cos B \cos C \leq 8\sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{B}{2}.$$

Using that $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ and $2\sin^2 \dfrac{A}{2} = \dfrac{a^2-(b-c)^2}{2bc}$ we get

$$\cos A \cos B \cos C \leq 8\sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}$$

$$\iff \prod_{cyc} \frac{b^2+c^2-a^2}{2bc} \leq \prod_{cyc} \frac{a^2-(b-c)^2}{2bc}$$

$$\iff \prod_{cyc} \left(b^2+c^2-a^2\right) \leq \prod_{cyc} (b+c-a)^2.$$

Without loss of generality we can assume that $\prod_{cyc} \left(b^2+c^2-a^2\right) > 0$.

Then $b^2+c^2 > a^2$, $c^2+a^2 > b^2, a^2+b^2 > c^2$ and, therefore,

$$\prod_{cyc} \left(b^2+c^2-a^2\right) \leq \prod_{cyc} (b+c-a)^2 \iff \prod_{cyc} \left(b^2+c^2-a^2\right)^2 \leq \prod_{cyc} (b+c-a)^4.$$

Because

$$\prod_{cyc}\left(b^2+c^2-a^2\right)^2=\prod_{cyc}\left(b^4-\left(c^2-a^2\right)^2\right)$$

and

$$\prod_{cyc}(b+c-a)^4=\prod_{cyc}\left(b^2-(c-a)^2\right)^2,$$

it is enough to prove $b^4-\left(c^2-a^2\right)^2\le\left(b^2-(c-a)^2\right)^2$. We have

$$\begin{aligned}\left(b^2-(c-a)^2\right)^2-b^4+\left(c^2-a^2\right)^2&=b^4-2b^2\left(c-a\right)^2+(c-a)^4-b^4+\left(c^2-a^2\right)^2\\&=(c-a)^2\left((c+a)^2-2b^2+(c-a)^2\right)\\&=(c-a)^2\left(2c^2+2a^2-2b^2\right)\\&=2\left(c-a\right)^2\left(c^2+a^2-b^2\right)\ge 0,\end{aligned}$$

and we are done.

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy