S549. Let a, b, c be positive real numbers such that $a + b + c + abc = 4$. Prove that

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \le \sqrt{1 + 4a - a^2} + \sqrt{1 + 4b - b^2} + \sqrt{1 + 4c - c^2} \le ab + bc + ca + 3$$

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First, note that in fact holds inequality

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} + 3 \le \sqrt{1 + 4a - a^2} + \sqrt{1 + 4b - b^2} + \sqrt{1 + 4c - c^2} \le ab + bc + ca + 3. \quad (1)$$

Indeed, since $2\sqrt{bc} \le b + c = 4 - a - abc \iff 2a\sqrt{bc} \le 4a - a^2 - a^2bc \iff$

$$1 + 2a\sqrt{bc} + a^2bc \le 1 + 4a - a^2 \iff 1 + a\sqrt{bc} \le \sqrt{1 + 4a - a^2}$$

and

$$a\left(\frac{b+c}{2}\right)^2 \ge abc = 4 - a - b - c = 4 - a - \frac{2(b+c)}{2} \iff a\left(\frac{b+c}{2}\right)^2 + \frac{2(b+c)}{2} \ge 4 - a \iff$$

$$a^2\left(\frac{b+c}{2}\right)^2 + 2a \cdot \frac{b+c}{2} \ge 4a - a^2 \iff \left(\frac{a(b+c)}{2} + 1\right)^2 \ge 4a - a^2 + 1 \iff$$

$$\frac{a(b+c)}{2} + 1 \ge \sqrt{1 + 4a - a^2}$$

then $1 + a\sqrt{bc} \le \sqrt{1 + 4a - a^2} \le \dfrac{a(b+c)}{2} + 1$ and, therefore,

$$\sum_{cyc}\left(1 + a\sqrt{bc}\right) \le \sum_{cyc}\sqrt{1 + 4a - a^2} \le \sum_{cyc}\left(\frac{a(b+c)}{2} + 1\right) \iff (1).$$

Equalities in (1) occurs iff $a = b = c = 1$.

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