S517. Let a, b, c be real numbers such that

$$a^3 + b^3 + c^3 - 1 = 3(a-1)(b-1)(c-1).$$

Prove that $a + b + c \leq 2$.

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We will use compact notations, namely let $s := a+b+c, p := ab+bc+ca, q := abc$.
Without loss of generality we can assume that $s > 1$. Since $a^3+b^3+c^3 = s^3+3q-3sp$ then

$$a^3 + b^3 + c^3 - 1 = 3(a-1)(b-1)(c-1)$$

becomes

$$s^3 + 3q - 3sp = 3(q - p + s - 1) \iff s^3 - 3s + 2 = 3p(s-1) \iff$$
$$(s+2)(s-1)^2 = 3p(s-1) \iff$$
$$(s+2)(s-1) = 3p.$$

Noting that $3p = 3(ab+bc+ca) \leq (a+b+c)^2 = s^2$ we obtain that

$$(s+2)(s-1) \leq s^2 \iff s \leq 2.$$

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