S515. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^6 \geq 27(a+2)(b+2)(c+2)$$

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Replacing $\left(\sqrt[3]{a}, \sqrt[3]{b}, \sqrt[3]{c}\right)$ with (a, b, c) results in an equivalent problem:

$$(a+b+c)^6 \geq 27(a^3+2)(b^3+2)(c^3+2) \text{ if } a,b,c>0 \text{ and } abc=1$$

or, after homogenization

$$(a+b+c)^6 abc \geq 27(a^3+2abc)(b^3+2abc)(c^3+2abc) \iff$$

$$(a+b+c)^6 \geq 27(a^2+2bc)(b^2+2ca)(c^2+2ab).$$

By AM-GM Inequality we have

$$(a+b+c)^6 = \left((a+b+c)^2\right)^3 = \left((a^2+2bc)+(b^2+2ca)+(c^2+2ab)\right)^3 \geq 27(a^2+2bc)(b^2+2ca)(c^2+2ab)$$

and we are done.

Also solved by Daniel Lasaosa, Pamplona, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Kevin Soto, Palacios Huarmey, Perú; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.