S491. Prove that in any acute triangle ABC the following inequality holds:

$$\frac{1}{\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)^2}+\frac{1}{\left(\cos\frac{B}{2}+\cos\frac{C}{2}\right)^2}+\frac{1}{\left(\cos\frac{C}{2}+\cos\frac{A}{2}\right)^2}\geq 1.$$

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Let

$$\alpha:=\frac{\pi-A}{2},\beta:=\frac{\pi-B}{2},\gamma:=\frac{\pi-C}{2}.$$

Then $\alpha,\beta,\gamma>0,\alpha+\beta+\gamma=\pi$,

$$\sum\frac{1}{\left(\cos\dfrac{A}{2}+\cos\dfrac{B}{2}\right)^2}=\sum\frac{1}{(\sin\alpha+\sin\beta)^2}$$

and original inequality becomes

$$\sum\frac{1}{(\sin\alpha+\sin\beta)^2}\geq 1. \tag{1}$$

Let ABC be some triangle with angles α,β,γ and correspondent side lengths a,b,c. (Don't mixe this triangle with original acute triangle). Also, let R,r and s be circumradius, inradius and semiperimeter of this triangle.

Then from (1) it wollows that

$$\sum\frac{1}{(a+b)^2}\geq\frac{1}{4R^2}.$$

Since by Cauchy Inequality

$$\sum\frac{1}{(a+b)^2}\geq\frac{9}{\sum(a+b)^2}=\frac{9}{2\left(a^2+b^2+c^2+ab+bc+ca\right)}$$

then remains to prove inequality

$$\frac{9}{2\left(a^2+b^2+c^2+ab+bc+ca\right)}\geq\frac{1}{4R^2}\iff\quad a^2+b^2+c^2+ab+bc+ca\leq 18R^2.$$

The latter inequality holds because $a^2+b^2+c^2\leq 9R^2$ and $ab+bc+ca\leq a^2+b^2+c^2$.

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