S459. Solve in real numbers the system of equations

$$|x^2-2| = \sqrt{y+2}$$
$$|y^2-2| = \sqrt{z+2}$$
$$|z^2-2| = \sqrt{x+2}$$

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First note that

$$\begin{cases} |x^2-2| = \sqrt{y+2} \\ |y^2-2| = \sqrt{z+2} \\ |z^2-2| = \sqrt{x+2} \end{cases} \iff \begin{cases} y = \left(x^2-2\right)^2-2 \\ z = \left(y^2-2\right)^2-2 \\ x = \left(z^2-2\right)^2-2 \end{cases}$$

Noting that $x, y, z \geq -2$ we consider two cases:

First Case: Let $x, y, z \in [-2, 2]$.Then denoting $t := \arccos \frac{x}{2}$ we obtain $x = 2\cos t, t \in [0, \pi]$,

$$y = \left(4\cos^2 t - 2\right)^2 - 2 = 4\cos^2 2t - 2 = 2\cos 4t, z = \left(4\cos^2 4t - 2\right)^2 - 2 = 2\cos 16t$$

and

$$x = \left(4\cos^2 16t - 2\right)^2 - 2 = 2\cos 64t.$$

Hence, for $t \in [0, \pi]$ we have $2\cos t = 2\cos 64t \iff$

$$\cos 64t - \cos t = 0 \iff \left[\begin{array}{l} \sin \frac{65t}{2} = 0 \\ \sin \frac{63t}{2} = 0 \end{array}\right. \iff \left[\begin{array}{l} t = \frac{\pi(2n+1)}{65}, 0 \leq n \leq 32 \\ t = \frac{\pi(2n+1)}{63}, 0 \leq n \leq 31 \end{array}\right. .$$

Thus,

$$(x, y, z) = \left(2\cos \frac{\pi(2n+1)}{65}, 2\cos \frac{4\pi(2n+1)}{65}, 2\cos \frac{16(2n+1)}{65}\right), n = 0, 1, .., 32$$

and

$$(x, y, z) = \left(2\cos \frac{\pi(2n+1)}{63}, 2\cos \frac{4\pi(2n+1)}{63}, 2\cos \frac{16(2n+1)}{63}\right), n = 0, 1, .., 31.$$

Second Case: Let $x, y, z \geq 2$.Then using representation $x = t + \frac{1}{t}, t > 0$ we obtain

$$y = \left(\left(t + \frac{1}{t}\right)^2 - 2\right)^2 - 2 = t^4 + \frac{1}{t^4}, z = \left(\left(t^4 + \frac{1}{t^4}\right)^2 - 2\right)^2 - 2 = t^{16} + \frac{1}{t^{16}},$$

$$x = \left(\left(t^{16} + \frac{1}{t^{16}}\right)^2 - 2\right)^2 - 2 = t^{64} + \frac{1}{t^{64}} = t + \frac{1}{t}.$$

Equation $t^{64} + \frac{1}{t^{64}} = t + \frac{1}{t}$ has only solution $t = 1$,because for any $t > 0$ and any natural $n > 1$ holds inequality $t^n + \frac{1}{t^n} \geq t + \frac{1}{t}$, where equality occurs iff $t = 1$.
Indeed, $t^n + \frac{1}{t^n} \geq t + \frac{1}{t} \iff t^{2n} - t^{n+1} - t^{n-1} + 1 \geq 0 \iff \left(t^{n-1} - 1\right)\left(t^{n+1} - 1\right) \geq 0$

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