S388. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{11a-6}{c}+\frac{11b-6}{a}+\frac{1ac-6}{b}\leq\frac{15}{abc}.$$

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Note that

$$\sum_{cyc}\frac{11a-6}{c}\leq\frac{15}{abc}\iff 11\left(a^2b+b^2c+c^2a\right)-6\left(ab+bc+ca\right)\leq 15\iff$$

$$11\left(a^2b+b^2c+c^2a\right)\leq 5\left(a^2+b^2+c^2\right)+6\left(ab+bc+ca\right).$$

Since

$$a^2b+b^2c+c^2a+abc\leq\frac{4}{27}\left(a+b+c\right)^3\iff a^2b+b^2c+c^2a\leq\frac{4}{27}\left(a+b+c\right)^3-abc$$

it suffices to prove

(1) $\quad 11\left(\frac{4}{27}\left(a+b+c\right)^3-abc\right)\leq 5\left(a^2+b^2+c^2\right)+6\left(ab+bc+ca\right)$

Since

$$5\left(a^2+b^2+c^2\right)+6\left(ab+bc+ca\right)=2\left(a^2+b^2+c^2\right)+3\left(a+b+c\right)^2=$$

$$\frac{2\left(a^2+b^2+c^2\right)+3\left(a+b+c\right)^2}{\sqrt{3}}\cdot\sqrt{a^2+b^2+c^2}$$

suffices to prove homogeneous inequality

$$11\left(\frac{4}{27}\left(a+b+c\right)^3-abc\right)\leq\frac{2\left(a^2+b^2+c^2\right)+3\left(a+b+c\right)^2}{\sqrt{3}}\cdot\sqrt{a^2+b^2+c^2}$$

which after normalization by $a+b+c=1$ becomes

$$11\sqrt{3}\left(\frac{4}{27}-q\right)\leq\left(2\left(1-2p\right)+3\right)\cdot\sqrt{1-2p}\iff$$

$$11\sqrt{3}\left(\frac{4}{27}-q\right)\leq\left(5-4p\right)\cdot\sqrt{1-2p}\ ,$$

where $p:=ab+bc+ca, q:=abc$ and

$q\geq q_*:=\max\left\{0,\frac{4p-1}{9}\right\},\ p\in(0,1/3]$.

Since $q\geq\frac{4p-1}{9}$ for $p\in(1/4,1/3]$ we consider inequality

$$11\sqrt{3}\left(\frac{4}{27}-q_*\right)=11\sqrt{3}\left(\frac{4}{27}-\frac{4p-1}{9}\right)\leq\left(5-4p\right)\cdot\sqrt{1-2p}\iff\left(5-4p\right)^2\left(1-2p\right)\geq$$

$$121\cdot\left(\frac{7-12p}{9\sqrt{3}}\right)^2\iff 243\left(5-4p\right)^2\left(1-2p\right)\geq 121\left(7-12p\right)^2.$$

We have $243\left(5-4p\right)^2\left(1-2p\right)-121\left(7-12p\right)^2=2\left(1-3p\right)\left(73-552p+1296p^2\right)\geq 0$
($p\in(1/4,1/3]$ and $73-552p+1296p^2\uparrow p\geq 1/4$ then $\min\left(73-552p+1296p^2\right)=$
$73-552\cdot\frac{1}{3}+1296\left(\frac{1}{3}\right)^2=33.0>0$).

Let $p \in (0, 1/4]$ then $11\sqrt{3}\left(\frac{4}{27} - q_*\right) = 11\sqrt{3}\left(\frac{4}{27} - 0\right) \leq (5-4p)\cdot\sqrt{1-2p}$. $\iff$

$$(5-4p)^2(1-2p) \geq 121\cdot\left(\frac{4}{9\sqrt{3}}\right)^2 \iff 243(5-4p)^2(1-2p) \geq 121\cdot 16$$

and $243(5-4p)^2(1-2p) - 121\cdot 16 = 4139 - 486\left(45p - 48p^2 + 16p^3\right)$
$\max\limits_{p\in(0,1/4]}\left(45p - 48p^2 + 16p^3\right)$
Since $\left(45p - 48p^2 + 16p^3\right)' = 45 - 96p + 48p^2 = 3(3-4p)(5-4p) > 0$ for $p \in (0, 1/4]$
then $\max\limits_{p\in(0,1/4]}\left(45p - 48p^2 + 16p^3\right) = 45\cdot\frac{1}{4} - 48\left(\frac{1}{4}\right)^2 + 16\left(\frac{1}{4}\right)^3 = \frac{17}{2}$ and, therefore,
$4139 - 486\left(45p - 48p^2 + 16p^3\right) \geq 4139 - 486\cdot\frac{17}{2} = 8 > 0.$

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