S357. Prove that in any triangle,

$$\sum \sqrt{\frac{a(h_a-2r)}{(3a+b+c)(h_a+2r)}} \leq \frac{3}{4}$$

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Since $h_a = \dfrac{2sr}{a}$, then

$$\frac{h_a-2r}{h_a+2r} = \frac{\dfrac{2sr}{a}-2r}{\dfrac{2sr}{a}+2r} = \frac{s-a}{s+a}\ ,$$

and by AM-GM,

$$\frac{a\left(h_a-2r\right)}{\left(3a+b+c\right)\left(h_a+2r\right)} = \frac{a\left(s-a\right)}{\left(2a+2s\right)\left(s+a\right)} = \frac{a\left(s-a\right)}{2\left(s+a\right)^2} = \frac{2a\left(s-a\right)}{4\left(s+a\right)^2} \leq \frac{\left(\dfrac{2a+\left(s-a\right)}{2}\right)^2}{4\left(s+a\right)^2} = \frac{1}{16}.$$

Hence, $\displaystyle\sum_{cyc} \sqrt{\frac{a\left(h_a-2r\right)}{\left(3a+b+c\right)\left(h_a+2r\right)}} \leq \sum_{cyc} \sqrt{\frac{1}{16}} = \frac{3}{4}.$

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