S317. Let ABC be an acute triangle inscribed in a circle of radius 1. Prove that

$$\frac{\tan A}{\tan^3 B}+\frac{\tan B}{\tan^3 C}+\frac{\tan C}{\tan^3 A}\geq 4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, California, USA

Since $a=2\sin A, b=2\sin B, c=2\sin C$ and

$$(\tan A-\tan B)\left(\frac{1}{\tan^3 A}-\frac{1}{\tan^3 B}\right)=-\frac{(\tan A-\tan B)^2\left(\tan^2 A+\tan A\tan B+\tan^2 B\right)}{\tan^3 A\tan^3 B}\leq 0$$

then by Rearrangement Inequality

$$\textstyle\sum_{cyc}\tan A\cdot\frac{1}{\tan^3 B}\geq\sum_{cyc}\tan A\cdot\frac{1}{\tan^3 A}=\sum_{cyc}\frac{1}{\tan^2 A}=\sum_{cyc}\cot^2 A=\sum_{cyc}\left(\frac{1}{\sin^2 A}-1\right)=$$

$$\sum_{cyc}\frac{1}{\sin^2 A}-3=4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-3.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Marius Stanean, Zalau, Romania; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Prithwijit De, HBCSE, Mumbai, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Li Zhou, Polk State College, USA.