S280. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$x^4y^4z^4(x^3+y^3+z^3) \leq 3.$$

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Since homogeneous form of original inequality is

(1) $3^{14}x^4y^4z^4(x^3+y^3+z^3) \leq (x+y+z)^{15}$

then assuming $x+y+z=1$ and denoting

$$t := 3\left(xy+yz+zx\right), q := xyz$$

we obtain (1) $\iff 3^{14}q^4\left(1-t+3q\right) \leq 1 \iff 1-3^{14}q^4\left(1-t\right)-3^{15}q^5 \geq 0.$

Since

$$3xyz\left(x+y+z\right) \leq \left(xy+yz+zx\right)^2 \iff q \leq \frac{t^2}{3^3} \Rightarrow$$

$$\Rightarrow 1-3^{14}q^4\left(1-3p\right)-3^{15}q^5 \geq 1-3^{14}\left(\frac{t^2}{3^3}\right)^4\left(1-t\right)-3^{15}\left(\frac{p^2}{3^3}\right)^5 =$$

$$1-9t^8\left(1-t\right)-t^{10} = \left(1-t\right)\left(t^9+\left(1+t+t^2+t^3+t^4+t^5+t^6+t^7-8t^8\right)\right) =$$

$$t^9\left(1-t\right)+\left(1-t\right)^2\left(1+2t+3t^2+4t^3+5t^4+6t^5+7t^6+8t^7\right) \geq 0$$

because

$$t = 3\left(xy+yz+zx\right) \leq \left(x+y+z\right)^2 = 1.$$

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