S252. Let a, b, c be positive real numbers. Prove that

$$a(b-c)^2+b(c-a)^2+c(a-b)^2+2abc \geq 2\frac{\sqrt{3(a^4b^4+b^4c^4+c^4a^4)}}{a+b+c}.$$

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Since $\left(a^2b^2+b^2c^2+c^2a^2\right)^2 \geq 3\left(a^4b^4+b^4c^4+c^4a^4\right)$, it suffices that

$$a(b-c)^2+b(c-a)^2+c(a-b)^2+2abc \geq \frac{2\left(a^2b^2+b^2c^2+c^2a^2\right)}{a+b+c}.$$

To see this, write

$$\begin{aligned} & (a+b+c)\left(a(b-c)^2+b(c-a)^2+c(a-b)^2+2abc\right)-2\left(a^2b^2+b^2c^2+c^2a^2\right) \\ = & (a+b+c)^2(ab+bc+ca)-7abc(a+b+c)-2\left((ab+bc+ca)^2-2abc(a+b+c)\right) \\ = & (a+b+c)^2(ab+bc+ca)-2(ab+bc+ca)^2-3abc(a+b+c). \end{aligned}$$

The latter rewrites as

$$(ab+bc+ca)\left((a+b+c)^2-3(ab+bc+ca)^2\right)+\left((ab+bc+ca)^2-3abc(a+b+c)\right) \geq 0,$$

where the positivity holds because

$$(a+b+c)^2 \geq 3(ab+bc+ca)^2 \text{ and } (ab+bc+ca)^2 \geq 3abc(a+b+c).$$

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