S233. In triangle ABC with $\angle C = 60^\circ$, let AA' and BB' be the angle bisectors of $\angle A$ and $\angle B$. Prove that

$$\frac{a+b}{A'B'} \leq \left(1+\frac{c}{a}\right)\left(1+\frac{c}{b}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA

Since $CA' = \dfrac{ab}{b+c}, CB' = \dfrac{ab}{a+c}$, and $\angle C = 60^\circ$, the Law of Cosines yields

$$A'B'^2 = \frac{a^2b^2}{(a+c)^2} + \frac{a^2b^2}{(b+c)^2} - \frac{a^2b^2}{(a+c)(b+c)},$$

and, therefore,

$$\frac{a+b}{A'B'} \leq \left(1+\frac{c}{a}\right)\left(1+\frac{c}{b}\right) \text{ which rewrites as } ab(a+b) \leq A'B'(a+c)(b+c)$$

becomes equivalent to proving that

$$a^2b^2(a+b)^2 \leq \left(\frac{a^2b^2}{(a+c)^2} + \frac{a^2b^2}{(b+c)^2} - \frac{a^2b^2}{(a+c)(b+c)}\right)(a+c)^2(b+c)^2,$$

i.e.

$$(a+b)^2 \leq (b+c)^2 + (a+c)^2 - (a+c)(b+c).$$

Since $c^2 = a^2 + b^2 - ab$ (the Law of Cosines), then

$$\begin{aligned} (b+c)^2 + (a+c)^2 - (a+c)(b+c) - (a+b)^2 &= bc + ac + c^2 - 3ab = bc + ac + a^2 + b^2 - 4ab \\ &= bc + ac - 2ab + (a-b)^2. \end{aligned}$$

But $c^2 = a^2 + b^2 - ab \geq ab,\ \ (b+c)^2 \geq 4ab$; whence $(b+c)^2 c^2 \geq 4a^2b^2$ i.e. $bc + ac \geq 2ab$, and thus

$$(bc + ac - 2ab) + (a-b)^2 \geq 0.$$

This completes the proof. □

Also solved by Ioan Viorel Codreanu, Satulung, Maramure, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.