S230. Let x, y, z be positive real numbers such that

$$xy + yz + zx \geq \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

Prove that $x + y + z \geq \sqrt{3}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA

Note that

$$xy + yz + zx \geq \frac{1}{\sqrt{x^2 + y^2 + z^2}} \text{ if and only if } x^2 + y^2 + z^2 \geq \frac{1}{(xy + yz + zx)^2},$$

which rewrites as

$$(x + y + z)^2 \geq 2(xy + yz + zx) + \frac{1}{(xy + yz + zx)^2}.$$

But the AM-GM Inequality yields

$$2(xy + yz + zx) + \frac{1}{(xy + yz + zx)^2} \geq 3\sqrt[3]{(xy + yz + zx)^2 \cdot \frac{1}{(xy + yz + zx)^2}} = 3,$$

thus, it follows that $(x + y + z)^2 \geq 3$, which yields $x + y + z \geq \sqrt{3}$. This completes the proof.

Also solved by Shamil Asgarli, University of British Columbia, Canada; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramure, Romania; Christopher Wiriawan, Jakarta, Indonesia; Marin Sandu and Mihai Sandu, Bucharest, Romania; Sayan Das, Kolkata, India; Paolo Perfetti, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Vazgen Mikayelyan.