S221. Let $ABC$ be a triangle with centroid $G$ and let $F$ be a point that minimizes the quantity $PA+PB+PC$ over all points $P$ lying in the plane of $ABC$. Prove that

$$FG \leq \min(AG, BG, CG).$$

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There are two cases. First, if $ABC$ has an obtuse angle greater than or equal to $120°$, say $\angle A$, then $F = A$, and since $\min\{AG, BG, CG\} = AG$, we have $FG = \min\{AG, BG, CG\}$. If all angles are less than $120°$, then $F$ lies in the interior of the triangle and $\angle AFB = \angle BFC = \angle CFA$.

Let $a$, $b$, $c$ be the sidelengths of triangle $ABC$, $F$ its area and $x$, $y$, $z$ the lengths of the segments $FA$, $FB$, and $FC$, respectively. Then

$$F = [BFC] + [CFA] + [AFB] = \frac{\sqrt{3}}{4}(xy + yz + zx),$$

yielding $xy + yz + zx = \dfrac{4F}{\sqrt{3}}$.

We have $a^2 = y^2 + z^2 + yz$, $b^2 = z^2 + x^2 + zx$, $c^2 = x^2 + y^2 + xy$, so

$$2\left(x^2 + y^2 + z^2\right) + xy + yz + zx = a^2 + b^2 + c^2,$$

which gives

$$x^2 + y^2 + z^2 = \frac{a^2 + b^2 + c^2 - \dfrac{4F}{\sqrt{3}}}{2} = \frac{3\left(a^2 + b^2 + c^2\right) - 4\sqrt{3}F}{6}.$$

However,

$$\sum_{cyc} GA^2 = \sum_{cyc}\left(\frac{2}{3}m_a\right)^2 = \frac{4}{9}\sum_{cyc}\frac{2\left(b^2 + c^2\right) - a^2}{4} = \frac{a^2 + b^2 + c^2}{3},$$

By Lagrange's Formula we get

$$\begin{aligned} FG^2 &= \frac{1}{3}\sum_{cyc}\left(FA^2 - GA^2\right) \\ &= \frac{1}{3}\left(x^2 + y^2 + z^2 - \frac{a^2 + b^2 + c^2}{3}\right) \\ &= \frac{1}{3}\left(\frac{3\left(a^2 + b^2 + c^2\right) - 4\sqrt{3}F}{6} - \frac{a^2 + b^2 + c^2}{3}\right) \\ &= \frac{a^2 + b^2 + c^2 - 4\sqrt{3}F}{18}. \end{aligned}$$

Thus we are left to show that

$$a^2 + b^2 + c^2 - 4\sqrt{3}F \leq 18\min\left\{AG^2, BG^2, CG^2\right\},$$

which is equivalent to showing the following system of inequalities

$$\left\{\begin{array}{l} a^2 + b^2 + c^2 - 4\sqrt{3}F \leq 4\left(b^2 + c^2\right) - 2a^2 \\ a^2 + b^2 + c^2 - 4\sqrt{3}F \leq 4\left(c^2 + a^2\right) - 2b^2 \\ a^2 + b^2 + c^2 - 4\sqrt{3}F \leq 4\left(a^2 + b^2\right) - 2c^2 \end{array}\right. \iff \left\{\begin{array}{l} -4F \leq \sqrt{3}\left(b^2 + c^2 - a^2\right) \\ -4F \leq \sqrt{3}\left(c^2 + a^2 - b^2\right) \\ -4F \leq \sqrt{3}\left(a^2 + b^2 - c^2\right) \end{array}\right.$$

Now, if triangle $ABC$ isn't obtuse then all three inequalities obviously holds. Thus it remains to consider the case, when one of angles is obtuse. Assume WLOG that $\angle A$ is such that $90^\circ < \angle A < 120^\circ$. In this case $-\dfrac{1}{2} < \cos A$ and $\frac{\sqrt{3}}{2} < \sin A$. Thus

$$\begin{aligned} a^2 - b^2 - c^2 = -2bc\cos A < bc \quad &\Longrightarrow \quad \frac{\sqrt{3}}{4}\left(a^2 - b - c^2\right) < \frac{\sqrt{3}}{4}bc < \frac{bc\sin A}{2} = F \\ &\Longrightarrow \quad -4F < \sqrt{3}\left(b^2 + c^2 - a^2\right). \end{aligned}$$

Since angles $\angle B$ and $\angle C$ are acute, we have $c^2 + a^2 - b^2 > 0$ and $a^2 + b^2 - c^2 > 0$. So two other inequalities clearly hold. This completes the proof.

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