S207. Let a, b, c be distinct nonzero real numbers such that $ab+bc+ca=3$ and $a+b+c \neq abc+\frac{2}{abc}$. Prove that

$$\left(\sum_{cyc} \frac{a(b-c)}{bc-1}\right) \cdot \left(\sum_{cyc} \frac{bc-1}{a(b-c)}\right)$$

is the square of an integer.

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We have

$$a+b+c \neq abc+\frac{2}{abc} \iff a^2b^2c^2 - abc(a+b+c)+2 \neq 0 \iff (ab-1)(bc-1)(ca-1) \neq 0.$$

By denoting $x=bc-1, y=ca-1, z=ab-1$, it suffices to show that for nonzero real numbers x, y, z such that $x+y+z=0$,

$$\left(\sum_{cyc} \frac{z-y}{x}\right) \cdot \left(\sum_{cyc} \frac{x}{z-y}\right)$$

is the square of an integer. We will prove that this quantity is equal to 9.

Note that

$$\sum_{cyc} \frac{z-y}{x} = -\frac{(y-x)(z-y)(x-z)}{xyz}$$

and

$$\sum_{cyc} \frac{x}{z-y} = \frac{1}{(y-x)(z-y)(x-z)} \sum_{cyc} x(y-x)(x-z)$$

so that

$$\left(\sum_{cyc} \frac{z-y}{x}\right) \cdot \left(\sum_{cyc} \frac{x}{z-y}\right) = \frac{\sum_{cyc} x(x-y)(x-z)}{xyz}.$$

Next,

$$\begin{aligned} \sum_{cyc} x(x-y)(x-z) &= \sum_{cyc} x\left(x^2-xy-xz+yz\right) \\ &= \sum_{cyc} x^3 - \sum_{cyc} x^2(y+z) + 3xyz. \end{aligned}$$

and the hypothesis $x+y+z=0$ yields $\sum_{cyc} x^3 = 3xyz$, $\sum_{cyc} x^2(y+z) = -\sum_{cyc} x^3 = -3xyz$. All in all,

$$\left(\sum_{cyc} \frac{z-y}{x}\right) \cdot \left(\sum_{cyc} \frac{x}{z-y}\right) = 9.$$

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