S184. Let $H_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \geq 2$. Prove that

$$e^{H_n} > \sqrt[n]{n!} \geq 2^{H_n}.$$

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LHS. By the AM-GM inequality we have

$$\sqrt[n]{n!} < \frac{1+2+\cdots+n}{n} = \frac{n+1}{2}.$$

Since

$$\left(1+\frac{1}{n}\right)^n < e \iff 1+\frac{1}{n} < e^{\frac{1}{n}} \iff \frac{n+1}{n} < e^{\frac{1}{n}}$$

for any positive integer n, then

$$\prod_{k=2}^{n} e^{\frac{1}{k}} > \prod_{k=2}^{n} \frac{k+1}{k} \iff e^{H_n} > \frac{n+1}{2} \implies e^{H_n} > \frac{n+1}{2} > \sqrt[n]{n!}.$$

RHS. Note that

$$n+1 > 2^{(n+1)H_{n+1} - nH_n}, n \geq 2.$$

Indeed, since

$$(n+1)\, H_{n+1} - nH_n = (n+1)\, H_n + 1 - nH_n = 1 + H_n$$

then

$$n+1 > 2^{(n+1)H_{n+1}-nH_n} \iff n+1 > 2^{1+H_n} \iff 2^{H_n} < \frac{n+1}{2}, n \geq 2. \quad (1)$$

We will prove inequality (1) by math induction.

1. Base case

If $n = 2$ then $2^{H_2} = \sqrt{2}$ and $\sqrt{2} < \frac{3}{2} \iff 8 < 9$.

2. Step case

Note that $\frac{n+2}{n+1} > 2^{\frac{1}{n+1}}$. Indeed, by the AM-GM Inequality

$$\sqrt[n+1]{2} = \sqrt[n+1]{2 \cdot 1 \cdot 1 \cdots 1} < \frac{2 + n \cdot 1}{n+1} = \frac{n+2}{n+1}.$$

Since $2^{\frac{1}{n+1}} < \frac{n+2}{n+1}$ and by supposition of math induction $2^{H_n} < \frac{n+1}{2}$, $n \geq 2$. Then

$$2^{H_n} \cdot 2^{\frac{1}{n+1}} < \frac{n+1}{2} \cdot \frac{n+2}{n+1} \iff 2^{H_{n+1}} < \frac{n+2}{2}.$$

And again by math induction, since for $n = 2$ we have $2! = 2^{2H_2}$ and from supposition $n! \geq 2^{nH_n}$ it follows that

$$\begin{aligned}(n+1)! &= (n+1)\, n! \geq (n+1)\, 2^{nH_n} \\ &> 2^{(n+1)H_{n+1}-nH_n} \cdot 2^{nH_n} = 2^{(n+1)H_{n+1}}.\end{aligned}$$

Then $n! \geq 2^{nH_n} \iff \sqrt[n]{n!} \geq 2^{H_n}$ for any $n \geq 2$.

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