S180. Solve in nonzero real numbers the system of equations

$$\begin{cases} x^4 - y^4 = \frac{121x-122y}{4xy} \\ x^4 + 14x^2y^2 + y^4 = \frac{122x+121y}{x^2+y^2}. \end{cases}$$

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Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that $x^4 + 14x^2y^2 + y^4 = 4(x^2+y^2)^2 - 3(x^2-y^2)^2 = s^4 - s^2d^2 + d^4$, where we have defined $x+y = s$ and $x-y = d$, while $x^4 - y^4 = sd(x^2+y^2) = \frac{sd(s^2+d^2)}{2}$, $4xy = s^2 - d^2$, $x^2 + y^2 = \frac{s^2+d^2}{2}$. Therefore, the system may be rewritten as

$$\begin{cases} sd(s^2+d^2)(s^2-d^2) = 243d - s, \\ (s^4 - s^2d^2 + d^4)(s^2+d^2) = 243s + d. \end{cases}$$

We can then obtain

$$(243d - s)(243s + d) = sd(s^2+d^2)(s^2-d^2)(243s+d) = (s^4 - s^2d^2 + d^4)(s^2+d^2)(243d - s).$$

Since $s^2 + d^2 > 0$ (otherwise $x = y = 0$, in contradiction with the problem statement), it follows that

$$243s^2d(s^2-d^2) + sd^2(s^2-d^2) = 243d(s^4 - s^2d^2 + d^4) - s(s^4 - s^2d^2 + d^4),$$

which after simplification yields $s^5 = 243d^5$, or $s = 3d$. Substitution in both equations yields $d^6 = d$, or since $x \neq y$ (if $x = y \neq 0$ the LHS of the first equation would be zero, but the RHS would not), we find that $d^5 = 1$, ie $s = 3$ and $d = 1$ for $x = 2$, $y = 1$. These values can be clearly shown to satisfy the system by plugging them into the given equations, and no other solutions exist.

Second solution by Arkady Alt, San Jose, California, USA

Since

$$\left(x^4 + 14x^2y^2 + y^4\right)\left(x^2+y^2\right) = 122x + 121y,\ 4xy\left(x^4 - y^4\right) = 121x - 122y$$

and $x, y \neq 0$ then

$$\begin{aligned} &\left(x^4 + 14x^2y^2 + y^4\right)\left(x^2+y^2\right)(x-y) - 4xy\left(x^4-y^4\right)(x+y) \\ &= (122x+121y)(x-y) - (121x-122y)(x+y) \\ &\iff \left(x^2+y^2\right)\left(\left(x^4+14x^2y^2+y^4\right)(x-y) - 4xy\left(x^2-y^2\right)(x+y)\right) \\ &= x^2+y^2 \iff \left(x^4+14x^2y^2+y^4\right)(x-y) - 4xy\left(x^2-y^2\right)(x+y) \\ &= 1 \iff (x-y)^5 = 1 \iff x - y = 1. \end{aligned}$$

Let $t = x + y$ then

$$x^2 - y^2 = t, x^2 + y^2 = \frac{t^2+1}{2},\ 4xy = t^2 - 1,\ y = \frac{t-1}{2},\ 121x - 122y = 121 - \frac{t-1}{2}$$

and the equation $x^4 - y^4 = \dfrac{121x - 122y}{4xy}$ becomes

$$\frac{t\left(t^4 - 1\right)}{2} = 121 - \frac{t-1}{2} \iff t\left(t^4 - 1\right) + t - 1 = 242 \iff t^5 = 243 \iff t = 3.$$

Hence,

$$\begin{cases} x - y = 1 \\ x + y = 3 \end{cases} \iff x = 2, y = 1.$$