S177. Prove that in any acute triangle ABC,

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\geq\frac{5R+2r}{4R}.$$

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Since $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, the inequality can be rewritten as

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq\frac{5}{4}. \qquad (A)$$

Inequality **(A)** is an immediate corollary from more general inequality represented by the following theorem

Theorem. *Let k be any real number such that $k\geq k_*$ where*

$$k_*=\frac{4}{2\sqrt{2-\sqrt{2}}-\sqrt{2}+3}\approx 1.2835.$$

Then for any $\alpha,\beta,\gamma\in\left(0,\frac{\pi}{4}\right]$, such that $\alpha+\beta+\gamma=\frac{\pi}{2}$ the following inequality holds

$$\sin\alpha+\sin\beta+\sin\gamma-k\sin\alpha\sin\beta\sin\gamma\geq\frac{12-k}{8}. \qquad (M)$$

Proof. Assuming, due symmetry, that $\alpha\leq\beta\leq\gamma$ and, denoting $\varphi=\alpha+\beta$ we obtain $\gamma=\frac{\pi}{2}-\varphi$, $\beta=\varphi-\alpha$, where the new variables α and φ satisfy the inequalities $0<\alpha\leq\varphi-\alpha\leq\frac{\pi}{2}-\varphi\leq\frac{\pi}{4}$ or equivalently

$$\begin{cases}\frac{\pi}{4}\leq\varphi\leq\frac{\pi}{3}\\ 2\varphi-\frac{\pi}{2}\leq\alpha\leq\frac{\varphi}{2}\end{cases}. \qquad (1)$$

Since

$$\begin{aligned}\sin\alpha+\sin\beta+\sin\gamma-k\sin\alpha\sin\beta\sin\gamma&=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\sin\gamma\left(1-k\ \sin\alpha\sin\beta\right)\\ &=2\sin\frac{\varphi}{2}\cos\left(\frac{\varphi}{2}-\alpha\right)+\cos\varphi\left(1-k\ \sin\alpha\sin\left(\varphi-\alpha\right)\right)\end{aligned}$$

then inequality **(M)** can be equivalently rewritten as

$$2\sin\frac{\varphi}{2}\cos\left(\frac{\varphi}{2}-\alpha\right)+\cos\varphi\left(1-k\ \sin\alpha\sin\left(\varphi-\alpha\right)\right)\geq\frac{12-k}{8} \qquad (2)$$

where variables α and φ are subject to the system **(1).**

Let

$$h(\alpha) = 2\sin\frac{\varphi}{2}\cos\left(\frac{\varphi}{2} - \alpha\right) + \cos\varphi\,(1 - k\ \sin\alpha\sin(\varphi - \alpha))$$

for any fixed $\varphi \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ and $k_* > \frac{2}{\sqrt{3}} = 1.1547$. We will prove that $h(\alpha)$ is decreasing on $\left[2\varphi - \frac{\pi}{2}, \frac{\varphi}{2}\right]$. Indeed,

$$h'(\alpha) = 2\sin\left(\frac{\varphi}{2} - \alpha\right)\left(\sin\frac{\varphi}{2} - k\cos\varphi\cos\left(\frac{\varphi}{2} - \alpha\right)\right) \le 0$$

on $\left[2\varphi - \frac{\pi}{2}, \frac{\varphi}{2}\right]$ since $\sin\left(\frac{\varphi}{2} - \alpha\right) \ge 0,\ k_* > \frac{2}{\sqrt{3}}$ and

$$\begin{aligned}\sin\frac{\varphi}{2} - k\cos\varphi\cos\left(\frac{\varphi}{2} - \alpha\right) &\le \sin\frac{\varphi}{2} - k\cos\varphi\cos\frac{\varphi}{2}\\ &\le \sin\frac{\pi}{6} - k\cos\frac{\pi}{3}\cos\frac{\pi}{6} = \frac{1}{2} - \frac{k}{2}\cdot\frac{\sqrt{3}}{2}\\ &\le \frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{k_*}{2} < \frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}}\\ &= 0.\end{aligned}$$

Thus, $h(\alpha) \ge h\left(\frac{\varphi}{2}\right) = 2\sin\frac{\varphi}{2} + \cos\varphi\left(1 - k\ \sin^2\frac{\varphi}{2}\right)$ and it remains to prove the inequality

$$2\sin\frac{\varphi}{2} + \cos\varphi\left(1 - k\sin^2\frac{\varphi}{2}\right) \ge \frac{12 - k}{8}. \tag{3}$$

Let $t = \sin\frac{\varphi}{2}$ then $\sin\frac{\pi}{8} \le t \le \frac{1}{2}$ and **(3)** is equivalent to

$$\begin{aligned}2t + \left(1 - 2t^2\right)\left(1 - kt^2\right) \ge \frac{12 - k}{8} &\iff 16kt^4 - (8k + 16)\,t^2 + 16t - (4 - k) \ge 0\\ &\iff (1 - 2t)^2\left(k\,(2t + 1)^2 - 4\right) \ge 0\end{aligned}$$

because $k\,(2t + 1)^2 \ge k_*\left(2\sin\frac{\pi}{8} + 1\right)^2 = 4$. Since equality in **(2)** occurs if and only if $\alpha = \frac{\varphi}{2}$ and $\varphi = \frac{\pi}{3} \iff \alpha = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$ then in **(M)** equality occurs if and only if $\alpha = \beta = \gamma = \frac{\pi}{6}$. In particular for $k = 2$ and $k = \frac{4}{3}$, replacing (α, β, γ) in **(M)** with $\left(\frac{A}{2}, \frac{B}{2}, \frac{C}{2}\right)$, for any acute triangle ABC we, respectively, obtain inequality **(A)** and inequality

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - \frac{4}{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \ge \frac{4}{3}. \tag{G}$$

(The last one is an inequality due to J. Garfuncel, given in **[RAGI]** without proof in a private communication.)

Remark.

The original inequality immediately follows from **(G).** Indeed,

$$\begin{aligned}\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-\frac{4}{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\ &\quad -\frac{2}{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\ &\geq \frac{4}{3}-\frac{2}{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq\frac{4}{3}-\frac{2}{3}\cdot\frac{1}{8}\\ &= \frac{5}{4}.\end{aligned}$$

[RAGI]. Mitrinović D.S., Pečarić J. E. , Volenec V. Recent Advances, *Geometric Inequality*, p.269, inequality 5.10.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume wlog that $C \geq B \geq A$, and denote $\frac{A+B}{4}=\alpha$, $\frac{B-A}{4}=\delta$. It is well known that $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, or the proposed problem is equivalent to showing that

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq\frac{5}{4}.$$

Assume that C is known such that the LHS is minimum, or for that value of C, define

$$f(x,y)=\sin x+\sin y-2\sin x\sin y\sin\frac{C}{2},$$

where $x+y=\frac{A+B}{2}=90^\circ-\frac{C}{2}$ is fixed. Now,

$$f\left(\frac{A}{2},\frac{B}{2}\right)-f(\alpha,\alpha)=f(\alpha-\delta,\alpha+\delta)-f(\alpha,\alpha)=2\sin\alpha(\cos\delta-1)+2\sin^2\delta\sin\frac{C}{2}=$$

$$=4\sin^2\frac{\delta}{2}\left(2\cos^2\frac{\delta}{2}\sin\frac{C}{2}-\sin\alpha\right).$$

Now, if $\delta>0$, and since $\delta=\frac{B-A}{4}<\frac{B}{4}<45^\circ$, we have $\sin^2\frac{\delta}{2}>0$, $2\cos^2\frac{\delta}{2}>2\cos^2(45^\circ)>1$. Moreover, $\frac{C}{2}>\frac{A+B}{4}$, since equality would only hold iff $A=B=C$, which is not true because $\delta>0$. Thus, $f\left(\frac{A}{2},\frac{B}{2}\right)\geq f\left(\frac{A+B}{4},\frac{A+B}{4}\right)$, with equality iff $A=B=90^\circ-\frac{C}{2}$. It therefore suffices to show that, for all $90^\circ>C\geq 60^\circ$, we have

$$2u+(1-2u^2)-2u^2(1-2u^2)\geq\frac{5}{4}.$$

where we have defined $u=\sin\left(45^\circ-\frac{C}{4}\right)$, and therefore $\sin\frac{C}{2}=\cos\left(2\left(45^\circ-\frac{C}{4}\right)\right)=1-2u^2$. After some algebra, this last inequality is equivalent to $(2u-1)^2(4u^2+8u+1)\geq 0$. Since $90^\circ>C\geq 60^\circ$, we have $\frac{45^\circ}{2}<45^\circ-\frac{C}{4}\leq 30^\circ$, or $u>0$. The conclusion follows, equality holds iff $u=\frac{1}{2}$, ie iff $45^\circ-\frac{C}{4}=30^\circ$, or $C=60^\circ$. We conclude that equality holds iff ABC is equilateral.

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