S174. Prove that for each positive integer k the equation

$$x_1^3 + x_2^3 + \cdots + x_k^3 + x_{k+1}^2 = x_{k+2}^4$$

has infinitely many solutions in positive integers with $x_1 < x_2 < \cdots < x_{k+1}$.

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First solution by Arkady Alt, San Jose, California, USA

Since

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2 (k+1)^2}{4} = (1 + 2 + \cdots + k)^2$$

then by substitution

$$x_1 = x, x_2 = 2x, \ldots, x_k = kx, x_{k+1} = (1 + 2 + \cdots + k)\, y^2, x_{k+2} = (1 + 2 + \cdots + k)\, y$$

in original equation, we obtain

$$\left(1^3 + 2^3 + \cdots + k^3\right) x^3 + (1 + 2 + \cdots + k)^2\, y^4 = (1 + 2 + \cdots + k)^4\, y^4 \iff x^3 = ay^4$$

where $a = (1 + 2 + \cdots + k)^2 - 1 = \dfrac{(k-1)(k+2)\left(k^2 + k + 2\right)}{4}$. Since $x = a^7 n^4,\ y = a^5 n^3$ for any positive integer n , then $x^3 = ay^4$. Hence

$$x_1 = a^7 n^4, x_2 = 2a^7 n^4, \ldots, x_k = ka^7 n^4, x_{k+1} = (1 + 2 + \cdots + k)\, a^{10} n^6, x_{k+2} = (1 + 2 + \cdots + k)\, a^5 n^3$$

for any positive integer n is a solution to the original equation and obviously, $x_1 < x_2 < \cdots < x_{k+1}$.

Second solution by the authors

For any positive integer n we have the well-known identity:

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 + \cdots + (n+k)^3 = \left(\frac{(n+k)(n+k+1)}{2}\right)^2,$$

that is

$$\left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 + \cdots + (n+k)^3 = \left(\frac{(n+k)(n+k+1)}{2}\right)^2.$$

Consider the positive integers n such that the triangular number $t_{n+k} = \frac{(n+k)(n+k+1)}{2}$ is a perfect squares. There are infinitely many such integers since the relation $t_{n+k} = u^2$ is equivalent to the Pell's equation $(2n + 2k + 1)^2 - 2u^2 = 1$. The fundamental solution to this Pell equation is $(3, 2)$, hence all these integers are given by the sequence (n_s), where

$$2n_s + 2k + 1 + u_s\sqrt{2} = (3 + 2\sqrt{2})^s,$$

for s big enough such that $n_s \geq 1$.