S166. If $a_1, a_2, \ldots, a_k \in (0,1)$, and k, n are integers such that $k > n \geq 1$, prove that the following inequality holds

$$\min\{a_1(1-a_2)^n, a_2(1-a_3)^n, \ldots, a_k(1-a_1)^n\} \leq \frac{n^n}{(n+1)^{n+1}}.$$

Proposed by Marin Bancos, North University of Baia Mare, Romania

First solution by Arkady Alt, San Jose, California, USA

Let $M = \min\left\{a_1\left(1-a_2\right)^n, a_2\left(1-a_3\right)^n, \ldots, a_k\left(1-a_1\right)^n\right\}$ and for any function $f(x,y)$ let

$$\sum_{cyc}^{k} f\left(a_1, a_2\right) = f\left(a_1, a_2\right) + f\left(a_2, a_3\right) + \cdots + f\left(a_k, a_1\right).$$

Since for any $x, y \in (0,1)$ by the AM–GM inequality

$$\sqrt[n+1]{nx\left(1-y\right)^n} \leq \frac{nx+n-ny}{n+1} = \frac{n\left(x-y\right)+n}{n+1}.$$

Then

$$\begin{aligned}
\sqrt[n+1]{M} &= \min\left\{\sqrt[n+1]{a_1\left(1-a_2\right)^n},\ \sqrt[n+1]{a_2\left(1-a_3\right)^n}, \ldots,\ \sqrt[n+1]{a_k\left(1-a_1\right)^n}\right\} \\
&\leq \frac{1}{k}\sum_{cyc}^{k} \sqrt[n+1]{a_1\left(1-a_2\right)^n} \\
&= \frac{1}{k\sqrt[n+1]{n}}\sum_{cyc}^{k} \sqrt[n+1]{na_1\left(1-a_2\right)^n} \\
&\leq \frac{1}{k\left(n+1\right)\sqrt[n+1]{n}}\sum_{cyc}^{k}\left(n\left(a_1-a_2\right)+n\right) \\
&= \frac{nk}{k\left(n+1\right)\sqrt[n+1]{n}} = \frac{n}{\left(n+1\right)\sqrt[n+1]{n}} \\
&= \frac{\sqrt[n+1]{n^n}}{(n+1)}.
\end{aligned}$$

Then $M \leq \dfrac{n^n}{\left(n+1\right)^{n+1}}$.

Second solution by the author

"Reductio ad absurdam"

Let's suppose that the inequality doesn't hold.

Therefore

$$a_1(1-a_2)^n > \frac{n^n}{(n+1)^{n+1}}$$