S154. Let $k \geq 2$ be an integer and let $n_1, \ldots, n_k$ be positive integers. Prove that there are no rational numbers $x_1, \ldots, x_k, y_1, \ldots, y_k$ such that

$$(x_1 + y_1\sqrt{2})^{2n_1} + \cdots + (x_k + y_k\sqrt{2})^{2n_k} = 5 + 4\sqrt{2}.$$

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First solution by Neacsu Adrian, Pitesti, Romania

We note that if $x, y \in Q$ such that

$$[1] : x + y\sqrt{2} = 5 + 4\sqrt{2}$$

then $x = 5$ and $y = 4$. Indeed, if $y \neq 4$, $\sqrt{2} = \frac{5-x}{y-4} \in Q$, contradiction. So $y = 4$ and from [1], $x = 5$. The given relation can be written as

$$\begin{aligned} &\sum_{i=1}^{k} \left\{ \binom{2n_i}{0} {x_i}^{2n_i} + \binom{2n_i}{2} {x_i}^{2n_i-2} 2{y_i}^2 + \cdots \right\} \\ &+ \sum_{i=1}^{k} \left\{ \binom{2n_i}{1} {x_i}^{2n_i-1} y_i + \binom{2n_i}{3} {x_i}^{2n_i-3} 2{y_i}^3 + \cdots \right\} \sqrt{2} \\ &= 5 + 4\sqrt{2}, X + Y\sqrt{2} = 5 + 4\sqrt{2}. \end{aligned}$$

From the above observation it follows $X = 5$, $Y = 4$ and: $X - Y\sqrt{2} = 5 - 4\sqrt{2}$, that can be written back as

$$\sum_{i=1}^{k} (x_i - y_i\sqrt{2})^{2n_i} = 5 - 4\sqrt{2}.$$

The left-hand side is a positive number and right-hand side is a negative one, contradiction.

Second solution by Arkady Alt, San Jose, California, USA

Consider the set $\mathbb{Q}\left(\sqrt{2}\right) = \left\{ x + y\sqrt{2} \vdots\, x, y \in \mathbb{Q} \right\}$ which is a quadratic extension of $\mathbb{Q}$) which is closed with respect to addition and multiplication. Note that the numbers 1 and $\sqrt{2}$ are linearly independent since $x + y\sqrt{2} = 0 \iff y = 0 \implies x = 0$, because otherwise $\sqrt{2} = -\dfrac{x}{y} \in \mathbb{Q}$. Therefore if $x_1, y_1, x_2, y_2 \in \mathbb{Q}$ then $x_1 + y_1\sqrt{2} = x_2 + y_2\sqrt{2} \iff \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$.

For any number $z = x + y\sqrt{2}$ from $\mathbb{Q}\left(\sqrt{2}\right)$ we consider the number $\overline{z} = x - y\sqrt{2}$ which we call the conjugate of z. For conjugation, due to linear independency, 1 and $\sqrt{2}$ satisfy the properties

1. $\overline{u+v} = \overline{u} + \overline{v}$, $u, v \in \mathbb{Q}\left(\sqrt{2}\right)$;

2. $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$, $u, v \in \mathbb{Q}\left(\sqrt{2}\right)$.

Since $\left(x + y\sqrt{2}\right)^n = a_n(x, y) + b_n(x, y)\sqrt{2}$ for any positive integer n, where $a_n(x, y)$ and $b_n(x, y)$ are polynomials with integer coefficients then for any rational x, y numbers $a_n = a_n(x, y)$ and

$b_n = b_n(x, y)$ are rational as well. Then

$$\left(x - y\sqrt{2}\right)^n = \overline{\left(x + y\sqrt{2}\right)^n} = \overline{a_n(x, y) + b_n(x, y)\sqrt{2}} = a_n(x, y) - b_n(x, y)\sqrt{2}.$$

Hence, $\left(x_1 - y_1\sqrt{2}\right)^{2n_1} + ... + \left(x_k - y_k\sqrt{2}\right)^{2n_k} = 5 - 4\sqrt{2}$ and, therefore,

$$\sum_{i=1}^{k}\left(x_i + y_i\sqrt{2}\right)^{2n_i}\sum_{i=1}^{k}\left(x_i - y_i\sqrt{2}\right)^{2n_i} = \left(5 + 4\sqrt{2}\right)\left(5 - 4\sqrt{2}\right) = 25 - 32 = -7,$$

which is a contradiction because the left hand side of equality is obviously positive.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.