S123. Prove that in any triangle with sidelenghts a, b, c the following inequality holds:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+\frac{(b+c-a)(c+a-b)(a+b-c)}{abc} \geq 7.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

First solution by Arkady Alt, San Jose, California, USA

Let $s=\frac{a+b+c}{2}$ is semiperimeter of a triangle with sidelengths a, b, c.Then,due to triangle inequalites $a, b, c<s$ and setting $x:=s-a, y:=s-b, z:=s-c$ we obtain

$a=y+z, b=z+x, c=x+y, s=x+y+z$, where $x, y, z>0$.

Thus, original inequality becomes

$\sum_{cyc} \frac{2x+y+z}{y+z}+\frac{8xyz}{(y+z)(z+x)(x+y)} \geq 7 \iff \sum_{cyc} \frac{s+x}{s-x}+\frac{8xyz}{(s-x)(s-y)(s-z)} \geq 7 \iff$

$\sum_{cyc}(s+x)(s-y)(s-z)+8xyz \geq 7(s-x)(s-y)(s-z) \iff$

$\sum_{cyc}(s+x)(sx+yz)+8xyz \geq 7(xy+yz+zx-xyz) \iff 2s^3-8s(xy+yz+zx)+18xyz \geq 0$

$(x+y+z)^3-4(x+y+z)(xy+yz+zx)+9xyz \geq 0 \iff$

$\sum_{cyc} x(x-y)(x-z) \geq 0$ **(Schure Inequality)**.

Second solution by Gheorghe Pupazan, Chisinau, Republic of Moldova

We make the well-known substitution $a=x+y, b=y+z$ and $c=z+x$, where $x, y, z>0$. The inequality becomes equivalent to:

$$\frac{2x+y+z}{y+z}+\frac{x+2y+z}{z+x}+\frac{x+y+2z}{x+y}+\frac{8xyz}{(x+y)(y+z)(z+x)} \geq 7.$$

After multiplying both sides by $(x+y)(y+z)(z+x)$, the inequality becomes equivalent with:

$$2(x^3+y^3+z^3)+6xyz \geq 2xyz \cdot \sum xy(x+y)$$

$$\iff 2\sum x(x-y)(x-z) \geq 0$$

which is just Schur's inequality.