S112. Let a, b, c be the side lengths and let s be the semiperimeter of a triangle ABC. Prove that

$$(s-c)^a(s-a)^b(s-b)^c \le \left(\frac{a}{2}\right)^a \left(\frac{b}{2}\right)^b \left(\frac{c}{2}\right)^c.$$

Proposed by Johan Gunardi, Jakarta, Indonesia

First solution by Arkady Alt, San Jose, California, USA

By the weighted AM-GM inequality we have

$$\left(\frac{s-c}{a}\right)^a \left(\frac{s-a}{b}\right)^b \left(\frac{s-b}{c}\right)^c \le \left(\frac{\frac{s-c}{a}\cdot a + \frac{s-a}{b}\cdot b + \frac{s-b}{c}\cdot c}{a+b+c}\right)^{a+b+c} =$$

$$\frac{1}{2^{a+b+c}} \iff (s-c)^a (s-a)^b (s-b)^c \le \left(\frac{a}{2}\right)^a \left(\frac{b}{2}\right)^b \left(\frac{c}{2}\right)^c.$$

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

If the triangle is degenerate, then the LHS is identically zero, while the RHS is non-negative, the inequality being trivially true. We need to consider thus only non-degenerate triangles. Dividing by the RHS and taking logarithm, the inequality may be rewritten in the following equivalent form:

$$a\log\left(1+\frac{b-c}{a}\right) + b\log\left(1+\frac{c-a}{b}\right) + c\log\left(1+\frac{a-b}{c}\right) \le 0.$$

Clearly, $g(x) = x - \log(1+x)$ is zero for $x=0$, while $\frac{dg(x)}{dx} = 1 - \frac{1}{1+x}$. Note that if $x > 0$, then $g(x)$ strictly increases, while if $x < 0$, then $g(x)$ strictly decreases, or the maximum of $g(x)$ is 0, occurring iff $x = 0$. As a consequence, $x \ge \log(1+x)$ for all $x > -1$, with equality iff $x = 0$. The triangular inequality guarantees that $-1 < \frac{b-c}{a} < 1$ for non-degenerate triangles, or $a\log\left(1+\frac{b-c}{a}\right) \le b - c$ with equality iff $b = c$. Adding this inequality to its cyclic permutations we obtain the proposed inequality. Equality holds if and only if, either $a = b = c$, or the triangle is degenerate with two equal sides and one side of length 0.

Third solution by Nguyen Manh Dung, Hanoi University of Science, Vietnam

The inequality is equivalent to

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \le a^a b^b c^c$$

It suffices to shows that

$$a\ln\frac{a+b-c}{a} + b\ln\frac{b+c-a}{b} + c\ln\frac{c+a-b}{c} \le 0$$