O86. The sequence $\{x_n\}$ is defined by $x_1 = 1$, $x_2 = 3$ and $x_{n+1} = 6x_n - x_{n-1}$ for all $n \geq 1$. Prove that $x_n + (-1)^n$ is a perfect square for all $n \geq 1$.

Proposed by Brian Bradie, Christopher Newport University, USA

First solution by Arkady Alt, San Jose, California, USA

First we will find solution of recurrence equation $a_{n+1} - 6a_n + a_{n-1} = 0, n \in \mathbb{N}$ with $a_0 = 3, a_1 = 1$ as initial conditions $(a_0 = 6a_1 - a_2 = 3)$.

Since $a_n = c_1 x_1^n + c_2 x_2^n$, $n \in \mathbb{N} \cup \{0\}$, where $x_1 = 3 + 2\sqrt{2} = \left(1 + \sqrt{2}\right)^2$ and $x_2 = 3 - 2\sqrt{2} = \left(1 - \sqrt{2}\right)^2$ are solutions of characteristic equation $x^2 - 6x + 1 = 0$, and $c_1 = \dfrac{3 - 2\sqrt{2}}{2}, c_2 = \dfrac{3 + 2\sqrt{2}}{2}$ are solution of the system

$$\begin{cases} a_0 = 3 = c_1 + c_2 \\ a_1 = 1 = c_1 x_1 + c_2 x_2 \end{cases}.$$

then $a_n = \dfrac{\left(3 + 2\sqrt{2}\right)^{n-1} + \left(3 - 2\sqrt{2}\right)^{n-1}}{2} = \dfrac{\left(\sqrt{2} + 1\right)^{2(n-1)} + \left(1 - \sqrt{2}\right)^{2(n-1)}}{2}$.

Thus, $a_n + (-1)^n = \dfrac{\left(1 + \sqrt{2}\right)^{2(n-1)} + \left(1 - \sqrt{2}\right)^{2(n-1)} + 2(-1)^n}{2} =$

$\dfrac{\left(1 + \sqrt{2}\right)^{2(n-1)} + \left(1 - \sqrt{2}\right)^{2(n-1)} - 2\left(1 + \sqrt{2}\right)^{n-1}\left(1 - \sqrt{2}\right)^{n-1}}{2} = t_n^2$, where

$t_n = \dfrac{\left(1 + \sqrt{2}\right)^{n-1} - \left(1 - \sqrt{2}\right)^{n-1}}{\sqrt{2}}$ all t_n are non-negative integers because satisfy $t_{n+1} - 2t_n - t_{n-1} = 0$ and $t_0 = 2, t_1 = 0$.

Second solution by Roberto Bosch Cabrera, Cuba

Let $y_n = x_n + (-1)^n$ for $n \geq 1$. Then $y_1 = 0,\ y_2 = 4,\ y_3 = 16$ and

$$y_{n+1} = 5y_n + 5y_{n-1} - y_{n-2}$$

for $n \geq 3$. Also, let $z_1 = 0,\ z_2 = 2$, and for $n \geq 3$, set $z_n = 2z_{n-1} + z_{n-2}$. We will show that $y_n = z_n^2$ for all n. This holds for $n = 1, 2, 3$ and assuming the claim for $y_1, ..., y_n$ we have

$$\begin{aligned} y_{n+1} &= 5z_n^2 + 5z_{n-1}^2 - z_{n-2}^2 \\ &= 5z_n^2 + 5z_{n-1}^2 - (z_n - 2z_{n-1})^2 \\ &= 4z_n^2 + 4z_{n-1}z_n + z_{n-1}^2 \\ &= (2z_n + z_{n-1})^2 = z_{n+1}^2 \end{aligned}$$