O509. Prove that for any positive real numbers a, b, c

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \frac{27\left(a^3+b^3+c^3\right)}{(a+b+c)^3}+\frac{21}{4}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Assuming $a+b+c=1$ (due homogeneity of the inequality) and denoting $p:=ab+bc+ca, q:=abc$ we obtain

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{27\left(a^3+b^3+c^3\right)}{(a+b+c)^3}-\frac{21}{4}=\frac{p}{q}-27(1+3q-3p)-\frac{21}{4}.$$

Noting that $p=ab+bc+ca \leq \frac{(a+b+c)^2}{3}=\frac{1}{3}$ then, denoting $t:=\sqrt{1-3p}$, we obtain $p=\frac{1-t^2}{3}$, where $t \in [0,1)$.

Since the criterion of solvability of Vieta's system

$$\left\{\begin{array}{c} a+b+c=1 \\ ab+bc+ca=p=\dfrac{1-t^2}{3} \\ abc=q \end{array}\right|$$

in real a, b, c is

$$\frac{(1-2t)(1+t)^2}{27} \leq q \leq \frac{(1+2t)(1-t)^2}{27},$$

then

$$\frac{p}{q}-27(1+3q-3p)-\frac{21}{4} \geq \frac{\frac{1-t^2}{3}}{\frac{(1+2t)(1-t)^2}{27}}-27\left(1+3\cdot\frac{(1+2t)(1-t)^2}{27}-3\cdot\frac{1-t^2}{3}\right)=$$

$$\frac{9(t+1)}{(1-t)(2t+1)}-3\left(2t^3+6t^2+1\right)-\frac{21}{4}=\frac{3\left(4t^3+14t^2+5t+1\right)(2t-1)^2}{4(2t+1)(1-t)} \geq 0$$

where equality occurs iff $t=\frac{1}{2} \iff p=\frac{1}{4},\ q=\frac{(1+2\cdot(1/2))(1-(1/2))^2}{27}=\frac{1}{54}$.

From cubic equation $x^3-x^2+\frac{1}{4}x-\frac{1}{54}=0 \iff \frac{1}{108}(3x-2)(6x-1)^2=0$, we obtain $a=b=\frac{1}{6}, c=\frac{2}{3}$.

Also solved by Albert Stadler, Herrliberg, Switzerland; An Nguyen Huu Bui, High School For The Gifted, VNUHCM, Ho Chi Minh City, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.