## Olympiad problems

O43. Let $a, b, c$ be positive real numbers. Prove that

$$\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}} \geq \sqrt{\frac{16(a+b+c)^3}{3(a+b)(b+c)(c+a)}}.$$

Proposed by Vo Quoc Ba Can, Can Tho University, Vietnam

*First solution by Arkady Alt, San Jose, California, USA*

Let us solve the following inequality

$$\sqrt{\frac{y+z}{x}}+\sqrt{\frac{z+x}{y}}+\sqrt{\frac{x+y}{z}} \geq \sqrt{\frac{16(x+y+z)^3}{3(x+y)(y+z)(z+x)}}.$$

Let $a=y+z, b=z+x$, $c=x+y$, and $s=x+y+z=\frac{a+b+c}{2}$. Observe that $a, b, c$ determine triangle $ABC$ with semiperimeter $s$, area $F$, and circumradius $R$. Using our notations we can rewrite our inequality

$$\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}} \geq \sqrt{\frac{16s^3}{3abc}} \iff$$

$$\sum_{cyc}\sqrt{\frac{(s-b)(s-c)}{bc}} \geq \frac{4}{\sqrt{3}}\cdot\frac{Fs}{abc}=\frac{1}{\sqrt{3}}\cdot\frac{s}{R}.$$

We know that $\frac{s}{R}=\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ and

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}, \sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}, \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Our inequality is equivalent to

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} \geq \frac{4}{\sqrt{3}}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Denote by $\alpha=\frac{\pi-A}{2}, \beta=\frac{\pi-B}{2}, \gamma=\frac{\pi-C}{2}$. Observe that $\alpha+\beta+\gamma=\pi$ and $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right)$. Consider now an acute-angled triangle $A'B'C'$ with $A'=\alpha, B'=\beta, C'=\gamma$ with the same notations $a, b, c, s, R, r$, for the lengths of sides, the semiperimeter, the circumradius, and the inradius, respectively. Our inequality can be rewritten as

$$\cos\alpha+\cos\beta+\cos\gamma \geq \frac{4}{\sqrt{3}}\sin\alpha\sin\beta\sin\gamma.$$

Using the identity $\cos\alpha+\cos\beta+\cos\gamma=\frac{R+r}{R}$ and Euler's Inequality $R\geq 2r$ we get $\cos\alpha+\cos\beta+\cos\gamma\geq\frac{3r}{R}$. Also we know $\sin\alpha\sin\beta\sin\gamma=\frac{abc}{8R^3}=\frac{4Rrs}{8R^3}=\frac{rs}{2R^2}$. Thus, it suffices to prove

$$\frac{4}{\sqrt{3}}\cdot\frac{rs}{2R^2}\leq\frac{3r}{R}\text{ or }2s\leq 3\sqrt{3}R,$$

that is clear from the famous fact $9R^2\geq a^2+b^2+c^2\geq\dfrac{(a+b+c)^2}{3}=\dfrac{4s^2}{3}$.

*Second solution by Kee-Wai Lau, Hong Kong, China*

Our inequality is homogeneous, therefore we can assume that $a+b+c=1$. Let us rewrite it in the following form

$$\frac{b+c}{\sqrt{a}}\sqrt{(c+a)(a+b)}+\frac{c+a}{\sqrt{b}}\sqrt{(a+b)(b+c)}+\frac{a+b}{\sqrt{c}}\sqrt{(b+c)(c+a)}\geq\frac{4\sqrt{3}}{3}.$$

We have

$$\frac{b+c}{\sqrt{a}}\sqrt{(c+a)(a+b)}=\left(\frac{1}{\sqrt{a}}-\sqrt{a}\right)\sqrt{(c+a)(a+b)}=\sqrt{1+\frac{bc}{a}}-\sqrt{a}\sqrt{a+bc}.$$

Using similar expressions for $\frac{c+a}{\sqrt{b}}\sqrt{(a+b)(b+c)}$ and $\frac{a+b}{\sqrt{c}}\sqrt{(b+c)(c+a)}$ we see that the left hand side is equal to $S_1-S_2$, where

$$S_1=\sqrt{1+\frac{ab}{c}}+\sqrt{1+\frac{bc}{a}}+\sqrt{1+\frac{ca}{b}}$$

and

$$S_2=\sqrt{a}\sqrt{a+bc}+\sqrt{b}\sqrt{b+ac}+\sqrt{c}\sqrt{c+ab}.$$

From the AM-GM inequality we have

$$ab+bc+ca\leq\frac{1}{3},\ a^2+b^2+c^2\geq\frac{1}{3},\ \frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq 1.$$

Let us prove that $S_1\geq 2\sqrt{3}$. Using the AM-GM inequality we have

$$\left(\frac{S_1}{3}\right)^6\geq\left(1+\frac{ab}{c}\right)\left(1+\frac{bc}{a}\right)\left(1+\frac{ca}{b}\right)=1+a^2+b^2+c^2+\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}+abc=$$

$$=1+\frac{1}{9}(a+b+c)^2+\frac{8}{9}(a^2+b^2+c^2)+\frac{26}{27}\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)+$$