O437. Let a, b, c the side-lengths of a triangle ABC. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \le \frac{2s^2}{27r^2}+1$$

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Using well known cyclic inequality $abc + a^2b + b^2c + c^2a \le \dfrac{4(a+b+c)^3}{27}$ we obtain

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} = \frac{ab^2+a^2c+bc^2}{abc} \le \frac{4(a+b+c)^3}{27abc} - 1.$$

Thus, suffice to prove inequality

$$\frac{4(a+b+c)^3}{27abc} \le \frac{2s^2}{27r^2}+2 \iff \frac{4\cdot 8s^3}{27\cdot 4Rrs} \le \frac{2s^2}{27r^2}+2 \iff \frac{4s^2}{27Rr} \le \frac{s^2}{27r^2}+1.$$

Since $16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2$ (Gerretsn's Inequalities) and $R \ge 2r$(Eulers Inequality) we have

$$\frac{s^2}{27r^2}+1-\frac{4s^2}{27Rr} = \frac{s^2}{27r^2}-\frac{2s^2}{27Rr}+1-\frac{2s^2}{27Rr} =$$

$$\frac{s^2(R-2r)}{27Rr^2} - \left(\frac{2s^2}{27Rr}-1\right) \ge$$

$$\frac{\left(16Rr-5r^2\right)(R-2r)}{27Rr^2} - \frac{2\left(4R^2+4Rr+3r^2\right)-27Rr}{27Rr} = \frac{\left(16Rr-5r^2\right)(R-2r)}{27Rr^2} - \frac{(R-2r)(8R-3r)}{27Rr} =$$

$$\frac{R-2r}{27Rr^2}\left(16Rr-5r^2-r(8R-3r)\right) = \frac{2r(R-2r)(4R-r)}{27Rr^2} \ge 0.$$

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