O383. Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{6c}+\frac{b+c}{6a}+\frac{c+a}{6b}+2 \geq \sqrt{\frac{a+b}{2c}}+\sqrt{\frac{b+c}{2a}}+\sqrt{\frac{c+a}{2b}}.$$

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Since $2+\sum_{cyc}\frac{a+b}{6c} \geq \sum_{cyc}\sqrt{\frac{a+b}{2c}} \iff \left(2+\sum_{cyc}\frac{a+b}{6c}\right)^2 \geq \sum_{cyc}\frac{a+b}{2c}+\sum_{cyc}\sqrt{\frac{(a+b)(b+c)}{ca}}$

and

$$\sqrt{\frac{(a+b)(b+c)}{ca}} \leq \frac{1}{2}\left(\frac{a+b}{a}+\frac{b+c}{c}\right)=1+\frac{1}{2}\left(\frac{b}{a}+\frac{b}{c}\right)$$

then

$$\sum_{cyc}\sqrt{\frac{(a+b)(b+c)}{ca}} \leq 3+\frac{1}{2}\sum_{cyc}\left(\frac{b}{a}+\frac{b}{c}\right)=3+\sum_{cyc}\frac{a+b}{2c}$$

and

$$\sum_{cyc}\frac{a+b}{2c}+\sum_{cyc}\sqrt{\frac{(a+b)(b+c)}{ca}} \leq 3+\sum_{cyc}\frac{a+b}{c}.$$

Thus, suffice to prove inequality

(1)

$$\left(2+\sum_{cyc}\frac{a+b}{6c}\right)^2 \geq 3+\sum_{cyc}\frac{a+b}{c}.$$

Let $t:=\sum_{cyc}\frac{a+b}{6c}$.Since

$$t+\frac{1}{2}=\frac{1}{6}\sum_{cyc}\left(\frac{a+b}{c}+1\right)=\frac{1}{6}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

and $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ then $t+\frac{1}{2} \geq \frac{9}{6} \iff t \geq 1$ and **(1)** becomes
$(2+t)^2 \geq 3+6t \iff (t-1)^2 \geq 0$.
Since $t=1 \iff a=b=c$ then equality in original inequality occurs iff $a=b=c$.

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