O336. Let a, b, c be positive distinct real numbers and let u, v, w be positive real numbers such that

$a+b+c=u+v+w$ and $(a^2-bc)r+(b^2-ac)s+(c^2-ab)t \geq 0$

for $(r,s,t)=(u,v,w),(r,s,t)=(v,w,u),(r,s,t)=(w,u,v)$. Prove that

$$x^a y^b z^c + x^b y^c z^a + x^c y^a z^b \geq x^u y^v z^w + x^w y^u z^v + x^v y^w z^u$$

for all nonnegative numbers x, y, z.

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By the weighted AM-GM inequality,

$$px^a y^b z^c + qx^b y^c z^a + rx^c y^a z^b \geq (p+q+r)\left(x^{pa+qb+rc} y^{pb+qc+ra} z^{pc+qa+rb}\right)^{\frac{1}{p+q+r}},$$

for any positive reals p, q, r.

We now search for p, q, r such that $\begin{matrix} pa+qb+rc=u \\ pb+qc+ra=v \\ pc+qa+rb=w \end{matrix}$ or in matrix form $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix}=\begin{pmatrix} u \\ v \\ w \end{pmatrix}$.

Since $a+b+c=u+v+w$ then adding all equations we obtain

$$(p+q+r)(a+b+c)=u+v+w \iff p+q+r=1.$$

Since $\det\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}=\sum_{cyc} a\left(bc-a^2\right)=-\sum_{cyc} a\left(a^2-bc\right)=-(a+b+c)\left(a^2+b^2+c^2\right)$

and $\det\begin{pmatrix} u & b & c \\ v & c & a \\ w & a & b \end{pmatrix}=\sum_{cyc} u\left(bc-a^2\right)=-\sum_{cyc} u\left(a^2-bc\right)$, then $p=\dfrac{\sum_{cyc} u\left(a^2-bc\right)}{(a+b+c)\left(a^2+b^2+c^2\right)}$,

and cyclically

$$q=\frac{\sum_{cyc} v\left(a^2-bc\right)}{(a+b+c)\left(a^2+b^2+c^2\right)}, r=\frac{\sum_{cyc} w\left(a^2-bc\right)}{(a+b+c)\left(a^2+b^2+c^2\right)},$$

where, we get p, q, r all positive due condition

$$(a^2-bc)r+(b^2-ac)s+(c^2-ab)t \geq 0$$

for $(r,s,t)=(u,v,w)$, $(r,s,t)=(v,w,u)$, $(r,s,t)=(w,u,v)$.

For the obtained p, q, rwe have

$$px^a y^b z^c + qx^b y^c z^a + rx^c y^a z^b \geq x^u y^v z^w,$$

and similarly

$$qx^a y^b z^c + rx^b y^c z^a + px^c y^a z^b \geq x^w y^u z^v, rx^a y^b z^c + px^b y^c z^a + qx^c y^a z^b \geq x^v y^w z^u.$$

Adding these inequalities and taking in account that $p+q+r=1$ we finally obtain

$$x^a y^b z^c + x^b y^c z^a + x^c y^a z^b \geq x^u y^v z^w + x^w y^u z^v + x^v y^w z^u.$$