Olympiad problems

O259. Solve in integers the equation $x^5 + 15xy + y^5 = 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA

Due to symmetry of equation we may assume that $x \geq y$.

Since in the case $x = y$ we get equation $2x^5 + 15x^2 = 1$ which obviously have no solution in integers then further we also can assume $x > y$.

Consider three cases.

1. $xy = 0$.Then we get solution $(x, y) = (1, 0)$;

2. $xy < 0$.Then due to supposition $x > y$ we have $x > 0,\ y < 0$ and by replacing y in equation with $-y$ we obtain equation $x^5 - 15xy - y^5 = 1,$ where $x,\ y \geq 1$ (because now x, y are positive integers).

Since $x^5 - y^5 = 15xy + 1 > 0$ then $x - y > 0 \iff x \geq y + 1$ yields

$x^5 - y^5 = (x - y)\left(x^4 + y^4 + xy\left(x^2 + y^2\right) + x^2y^2\right) \geq 5(x - y)x^2y^2$ and $xy \geq 2$.Therefore,

$15xy + 1 \geq 5(x - y)x^2y^2 \iff 3xy + \dfrac{1}{3} \geq (x - y)x^2y^2 \implies 3 \geq (x - y)xy \iff \dfrac{3}{xy} \geq x - y$ and since $1 \leq x - y$ then $x - y = 1$ and $xy \leq 3$.

But system $\begin{cases} x - y = 1 \\ xy = 3 \end{cases}$ have no integer solutions. Then remains system $\begin{cases} x - y = 1 \\ xy = 2 \end{cases}$

which give us solution $x = 2, y = 1$.Since $(x, y) = (2, 1)$ satisfy $x^5 - 15xy - y^5 = 1$ then $(x, y) = (2, -1), (-1, 2)$ are solutions of original equation in the case $xy < 0$.

3. Let $xy > 0$. It is possible if $x, y < 0$.Then by replacing (x, y) in original equation with $(-x, -y)$ we obtain equation $x^5 + y^5 = 15xy - 1$ where $1 \leq x < y$.

Hence, $xy \geq 2$ and $x + y \geq 2x + 1 \geq 3$.Since $x^5 + y^5 = (x + y)\left(x^4 + y^4 - xy\left(x^2 + y^2\right) + x^2y^2\right) =$
$(x + y)\left(\left(xy + x^2 + y^2\right)(x - y)^2 + x^2y^2\right) \geq (x + y)\left(\left(xy + x^2 + y^2\right) + x^2y^2\right) \geq (x + y)\left(3xy + x^2y^2\right) =$
$(x + y)(3 + xy)xy \geq 3 \cdot (3 + 2)xy = 15xy$ then $15xy - 1 = x^5 + y^5 \geq 15xy$,that is the contradiction.

Thus, all solutions of original equation are $(x, y) = (1, 0), (0, 1), (2, -1), (-1, 2)$.

Also solved by Li Zhou, Polk State College, USA; George Batzolis, Mandoulides High School, Thessaloniki, Greece; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Semchankau Aliaksei, Minsk, Belarus; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy.